An Introduction to Function Fields

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References

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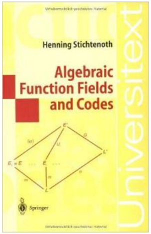

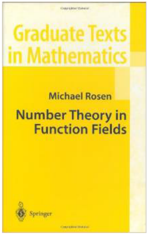

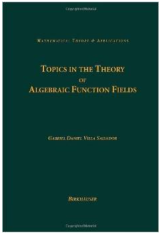

Valuation Theory

Absolute Values

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Definition

An absolute value on F is a map $|\cdot| : F \to \mathbb{R}$ such that for all $a, b \in F$:

- $|a| \geq 0$, with equality if and only if $a = 0$
- $|ab| = |a||b|$
- $|a + b| \leq |a| + |b|$ (archimedean) or
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Examples

- The well-known absolute value on $\mathbb{Q}$ (or on $\mathbb{R}$ or on $\mathbb{C}$) is an archimedean absolute value in the sense of the above definition.
- The trivial absolute value on any field F, defined via $|a| = 0$ when $a = 0$ and $|a| = 1$ otherwise, is a non-archimedean absolute value.

p-Adic Absolute Values on $\mathbb{Q}$

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Theorem (Ostrowski)

The p-adic absolute values, along with the trivial and the ordinary absolute value, are the only valuations on $\mathbb{Q}$.

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Note that $F = K(x)$ is our first example of an algebraic function field. More formally:

Definition

A rational function field F/K is a field F of the form $F = K(x)$ where $x \in F$ is transcendental over K.

Absolute Values on $K(x)$

Fix a constant $c \in \mathbb{R}$, $c > 1$, and let $r(x) \in K(x)$ be nonzero.

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Infinite absolute value on $K(x)$:
Write $r(x) = f(x)/g(x)$ and define

$$|r(x)|_\infty = c^{\deg(f)-\deg(g)}.$$

Then $|\cdot|_\infty$ is a non-archimedian absolute value on $K(x)$.

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- When $K = \mathbb{F}_q$ is a finite field of order q, one usually chooses $c = q$.
- When K is a field of characteristic 0, one usually chooses $c = e = 2.71828\ldots$.

Valuations

Definition

A valuation on F is a map $v : F \to \mathbb{R} \cup \{\infty\}$ such that for all $a, b \in F$:

- $v(a) = \infty$ if and only if $a = 0$
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Remark

Let $c > 1$ be any constant. Then v is a valuation on F if and only if $|\cdot| := c^{-v(\cdot)}$ is a non-archimedian absolute value on F (with $c^{-\infty} := 0$).

Examples

- **Trivial valuation**: for any $a \in F$, define $v(a) = \infty$ when $a = 0$ and $v(a) = 0$ otherwise. Then v is a valuation on F.

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- **Infinite valuation on** $K(x)$: for $r(x) = f(x)/g(x) \in K(x)^*$, define $v_\infty(r(x)) = \deg(g) - \deg(f)$. Then v_∞ is a valuation on $K(x)$.

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Remark

A discrete valuation is normalized if and only if it is surjective.

Valuation Rings

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Properties:

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- Every $a \in F^*$ has a unique representation $a = \epsilon u^n$ with $\epsilon \in O_v^*$ and $n = v(a) \in \mathbb{Z}$.
- O_v is principal ideal domain whose ideals are generated by the non-negative powers of u; in particular, u is a generator of P_v.

Example: p-Adic Valuations

For any p-adic valuation v_p on $\mathbb{Q}$:

$$\begin{aligned}
O_{v_p} &= \{r \in \mathbb{Q} \mid r = a/b \text{ with } \gcd(a,b) = 1 \text{ and } p \nmid b\} \\
O_{v_p}^* &= \{r \in \mathbb{Q} \mid r = a/b \text{ with } \gcd(a,b) = 1 \text{ and } p \nmid ab\} \\
P_{v_p} &= \{r \in \mathbb{Q} \mid r = a/b \text{ with } \gcd(a,b) = 1,\ p \mid a,\ p \nmid b\} \\
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Similarly, for any p-adic valuation $v_{p(x)}$ on $K(x)$:

$$\begin{aligned}
O_{v_{p(x)}} &= \{r(x) \in K(x) \mid r(x) = a(x)/b(x) \text{ with } \gcd(a,b) = 1,\ p(x) \nmid b(x)\} \\
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&\qquad\qquad p(x) \mid a(x),\ p(x) \nmid b(x)\} \\
F_{v_{p(x)}} &= K[x]/(p(x)) \text{ where } (p(x)) \text{ is the } K[x]\text{-ideal generated by } p(x)
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Example: Infinite Valuation on $K(x)$

For the infinite valuation v_∞ on $K(x)$:

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P_{v_\infty} &= \{r(x) \in K(x) \mid (x) = f(x)/g(x) \text{ with } \deg(f) < \deg(g)\} \\
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We will henceforth write O_∞, P_∞, F_∞ for brevity.

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Examples of Places

For any prime number p, the set

$$P = \{r \in \mathbb{Q} \mid r = a/b \text{ with } \gcd(a, b) = 1,\ p \mid a,\ p \nmid b\} = P_{v_p}$$

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The set $\mathbb{P}(K(x))$ consists of the finite places of $K(x)$ of the form $P_{p(x)} = P_{v_{p(x)}}$ where $p(x)$ is a monic irreducible polynomial in $K[x]$ and the infinite place of $K(x)$ of the form $P_\infty = P_{v_\infty}$.

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Let F be a finite algebraic extension of $\mathbb{F}_q(x)$ and let $\mathcal{O}_F$ be the integral closure of the polynomial ring $\mathbb{F}_q[x]$ in F. Then every prime ideal in $\mathcal{O}_F$ is a place of K.

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Function Fields

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In other words, a function field is of the form $F = K(x, y)$ where

- $x \in F$ is transcendental over K,
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Remark

It is important to note that there are many choices for x, and the degree $[F : K(x)]$ may change with the choice of x. This is different from number fields where the degree over $\mathbb{Q}$ is fixed.

A function field is rational if $F = K(x)$ for some element $x \in F$ that is transcendental over K.

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More generally, consider the curve $y^2 = f(x)$ where $f(x) \in K[x]$ is a square-free polynomial and K has characteristic different from 2. Then $F = K(x, y)$ is a function field over K whose elements are of the form

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The coordinate ring of a curve $C : \Phi(x, y) = 0$ over a field K is the ring $K[x, Y]/(\Phi(x, Y))$ where $(\Phi(x, Y))$ is the principal $K[x, Y]$-ideal generated by $\Phi(x, Y)$.

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Remark: The function field of a curve is a function field as defined previously. Conversely, every function field F/K is the function field of the curve given by a minimal polynomial of $F/K(x)$.

More on Function Fields and Curves

General form of a function field F/K:

$$F = K(x, y) \quad \text{with} \quad \Phi(x, y) = 0 \ ,$$

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Example: Let $A, B \in K$ and consider the two curves

$$\begin{aligned} C_1 &: y^2 = x^3 + Ax + B \ , \\ C_2 &: v^2 = Bu^4 + Au^3 + u \ . \end{aligned}$$

Then $K(x, y) = K(u, v)$.

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Then $K(x, y) = K(u, v)$.
Dividing C_1 by x^4 and putting $u = x^{-1}$, $v = yx^{-2}$ yields C_2.

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The constant field of a function field F/K is the algebraic closure of K in F, i.e. the field

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Remark

Write $F = K(x, y)$. Then F/K is a geometric function field if and only if the minimal polynomial of y over $K(x)$ is absolutely irreducible, i.e. irreducible over $\overline{K}(x)$ where $\overline{K}$ is the algebraic closure of K.

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  Over $\tilde{K}$, we have $x \pm iy^2 = 0$.

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- $K(x)/K$ is always geometric.
- If K is algebraically closed (e.g. $K = \mathbb{C}$), then any F/K is geometric.
- Let $F = K(x, y)$ where $y^2 = f(x)$ with $f(x) \in K[x]$ square-free. Then $F/K(x)$ is geometric if and only if $f(x)$ is non-constant (otherwise $\tilde{K} = K(y)$ and $F = \tilde{K}(x)$).
- Suppose -1 is not a square in K (e.g. $K = \mathbb{R}$ or $K = \mathbb{F}_q$ with $q \equiv 3 \pmod 4$).
  Let $F = K(x, y)$ where $x^2 + y^4 = 0$. Then $[F : K(x)] = 4$.
  Let $i \notin K$ be a square root of -1. Then $i^2 + 1 = 0$, so i is algebraic over K. Thus $i \in \tilde{K} \setminus K$. In fact, $\tilde{K} = K(i)$, so F/K is not geometric.
  Over $\tilde{K}$, we have $x \pm iy^2 = 0$.
  Note that $[\tilde{K} : K] = [\tilde{K}(x) : K(x)] = 2$ and $[F : \tilde{K}(x)] = 2$.

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Recall that a place P of a field F is the unique maximal ideal of some discrete valuation ring O_P of F, and its residue field is $F_P = O_P/P$.

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Remark

$\deg(P) \leq [F : K(x)]$ for any $x \in P$, so $\deg(P)$ is always finite.

Example: Residue Fields of Places of $K(x)$

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- For any finite place $P_{p(x)}$ of $K(x)$, a $K-$basis of F_P is $\{1, x, \ldots, x^{\deg(p)-1}\}$, so $\deg(P_{p(x)}) = \deg(p)$.

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  In this case, there is a one-to-one correspondence between $\mathbb{P}_1(K(x))$ and the *points on the projective line* $\mathbb{P}^1(K) := K \cup \{\infty\}$ via

  $$\mathbb{P}_1(K(x)) \longleftrightarrow \mathbb{P}^1(K) \quad \text{via} \quad x + \alpha \longleftrightarrow \alpha \ , \quad 1/x \longleftrightarrow \infty \ .$$

  Hence the name 'infinite place" — think of this as "substituting $x = 0$" into the uniformizer.

Divisors and Class Groups

Recollection: Ideals in Number Fields

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We now consider analogous notions in function fields, with prime ideals replaced by places, and multiplication (products) replaced by addition (sums).

Assume henceforth that F/K is a geometric function field.

Divisors

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- D is a prime divisor if it is of the form $D = P$ for some $P \in \mathbb{P}(F)$.

More on Divisors

Remarks

- Every divisor is a unique sum of finitely many prime divisors (note that some prime divisors in the support may have negative coefficients).

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- **F. K. Schmidt** proved that every function field F over a *finite* field $K = \mathbb{F}_q$ has a divisor of degree one, so in this case, the degree homomorphism on $\mathrm{Div}(F)$ is surjective.

Principal Divisors

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The zero divisor and pole divisor of a principal divisor $\mathrm{div}(z)$ are the respective divisors

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So $\mathrm{div}(z) = \mathrm{div}(z)_0 - \mathrm{div}(z)_\infty$.

Example: In $F = K(x)$, we have $\mathrm{div}(x)_0 = P_x$ and $\mathrm{div}(x)_\infty = P_\infty$.

More on Principal Divisors

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Remark and Notation

Linear equivalence is an equivalence relation. The class of a divisor D under linear equivalence is denoted $[D]$.

Class Group and Zero Class Group

Definition

The factor groups

$$\mathrm{Cl}(F) = \mathrm{Div}(F)/\,\mathrm{Prin}(F) \quad \text{and} \quad \mathrm{Cl}^0(F) = \mathrm{Div}^0(F)/\,\mathrm{Prin}(F)$$

are the divisor class group and the degree zero divisor class group of F/K, respectively.

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Let F/K be a non-rational function field that has a rational place, denoted P_∞. Then the map

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Rational Places and the Class Group

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The class group and class number are important invariants of any function field. Unfortunately, they are not easy to compute ... ☹

Effective Divisors

Definition

Define a partial order $\geq$ on $\mathrm{Div}(F)$ via

$$D_1 \geq D_2 \quad \Leftrightarrow \quad v_P(D_1) \geq v_P(D_2) \text{ for all } P \in \mathbb{P}(F).$$

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- The sum of two effective divisors is effective. So the effective divisors form a sub-monoid of $\mathrm{Div}(F)$.

Decomposition of Places

Recollection: Prime Ideals in Number Fields

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Recall that in a number field $F/\mathbb{Q}$:

- A prime $p \in \mathbb{Z}$ need not remain a prime (ideal) when extended to $\mathcal{O}_F$.

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- The fundamental identity $\sum_{i=1}^{r} e_i f_i = [F:\mathbb{Q}]$ holds.

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Once again, we consider analogous notions in function field extensions, with prime ideals replaced by places, and products replaced by sums.

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  Then $\phi(T) = (T^p - \alpha^p) = (T - \alpha)^p$, so α has multiplicity p.

Recap: the Places of $K(x)$

Finite places of $K(x)$:

- $P_{p(x)}$, where $p(x) \in K[x]$ is monic and irreducible;
- Uniformizer is $p(x)$;
- Residue field is $F_{P_{p(x)}} = K[x]/(p(x))$;
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Infinite place of $K(x)$:

- P_∞, corresponding to the infinite valuation (denominator degree minus numerator degree);
- Uniformizer is x^{-1};
- Residue field is $F_{P_\infty} = K$;
- Degree of P_∞ is $\deg(P_\infty) = 1$.

Places in $K(x)$ and F

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The "lift" $P\,O_{P'}$ of P to F is no longer a place. Rather, it is a divisor of F called the *co-norm* of P.

Decomposition Data

Theorem and Definition

- The co-norm of $P \in \mathbb{P}(K(x))$ is the divisor
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- $\deg(P') = f(P'|P)\deg(P)$ for all $P'|P$.
- Fundamental identity: $\sum_{P'|P} e(P'|P)f(P'|P) = n$ for all $P \in \mathbb{P}(K(x))$.

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Sufficient (but not necessary) conditions for a function field to be tamely ramified are:

- $\mathrm{char}(K) = 0$.
- $n < \mathrm{char}(K)$ when $\mathrm{char}(K)$ is positive.

Computing Ramification Data

Theorem (**Kummer's Theorem in function fields**)

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Two sufficient conditions for item 3 are:

- *All $\epsilon_i = 1$ (so $\Phi(Y)$ is squarefree modulo P)*

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3. *Under certain conditions, equality holds in items 1 and 2, and $e(P_i'|P) = \epsilon_i$.*

Two sufficient conditions for item 3 are:

- *All $\epsilon_i = 1$ (so $\Phi(Y)$ is squarefree modulo P) or*
- *$\{1, y, \ldots, y^{n-1}\}$ is an O_P-basis of $\bigcap_{i=1}^{r} O_{P_i'}$.*

Example: Quadratic Fields, Part I

Let $\mathrm{char}(K) \neq 2$, $F = K(x, y)$ where $x \in F$ is transcendental over K and $y^2 = f(x)$ with $f(x) \in K[x]$ square-free.

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For the infinite place $P = P_\infty$, recall that

- x^{-1} is a uniformizer of P_∞,
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Write $f(x) = ax^{2m-\delta} +$ terms of lower degree in x, with $0 \neq a \in K$ and $\delta \in \{0, 1\}$, and put $z = yx^{-m}$. Then

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Hence, 2 divides $e(P'|P)$, so P is totally ramified in F.

Explicit Example

Let $F = \mathbb{F}_5(x, y)$ with $y^2 = x^3 + x = x(x+2)(x+3) \in \mathbb{F}_5[x]$.

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- The ramified places of $\mathbb{F}_5(x)$ are P_x, P_{x+2}, P_{x+3} and P_∞.

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Remark: When $K = \mathbb{F}_q$, determining whether or not $f(x)$ is a square modulo $p(x)$ can be done with the quadratic residue symbol

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This function field version of the Legendre symbol can be computed via

$$\left(\frac{f(x)}{p(x)}\right) \equiv f(x)^{\frac{|p(x)|-1}{2}} \equiv f(x)^{\frac{q^{\deg(p(x))}-1}{2}} \pmod{p(x)} .$$

The Different

Assume that all places of $K(x)$ are tamely ramified in F.

Definition

The different (or ramification divisor) of $F/K(x)$ is

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for any $x \in F \setminus K$, where $n = [F : K(x)]$.

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Let $F/\mathbb{F}_q$ be a function field of genus g over a finite field of order q. Then

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Remark

There are 8 non-rational function fields $F/\mathbb{F}_q$ of class number one. All have $q \leq 4$, and defining curves for all of them are known.

Genus 0 and 1 Function Fields

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$F = \mathbb{R}(x, y)$ where $x^2 + y^2 = -1$ has genus 0 but is not rational.

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for some $a_1, a_2, a_3, a_4, a_6 \in K$. This equation defines an elliptic curve in Weierstraß form. Note that $[F : K(x)] = 2$ and $[F : K(y)] = 3$.

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- Similarly, if $\mathrm{char}(K) = 2$, one can always convert a (long) Weierstraß form to an equation of the form

$y^2 + y =$ cubic polynomial in x or $y^2 + xy =$ cubic polynomial in x.

$\mathbb{P}_1(F)$ as an Abelian Group

Theorem

Let F/K be an elliptic function field, and fix a rational place $P_\infty \in \mathbb{P}_1(F)$. Then the injection $\Phi : \mathbb{P}_1(F) \to \mathrm{Cl}^0(F)$ via $P \mapsto [P - P_\infty]$ is a bijection.

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- Every degree zero divisor class of F/K has a unique representative of the form $[P - P_\infty]$ with $P \in \mathbb{P}_1(F)$.
- The set $\mathbb{P}_1(F)$ becomes an abelian group (and Φ a group isomorphism) under the addition law

$$P \oplus Q =: R \quad \Longleftrightarrow \quad [P - P_\infty] + [Q - P_\infty] = [R - P_\infty].$$

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$$E_H : y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3.$$

Points on E_H: $[x : y : z] \neq [0 : 0 : 0]$ normalized to last non-zero entry $= 1$.

Points on E	$\longleftrightarrow$	Points on E_H
(x, y)	$\longrightarrow$	$[x : y : 1]$

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Points on an Elliptic Curve

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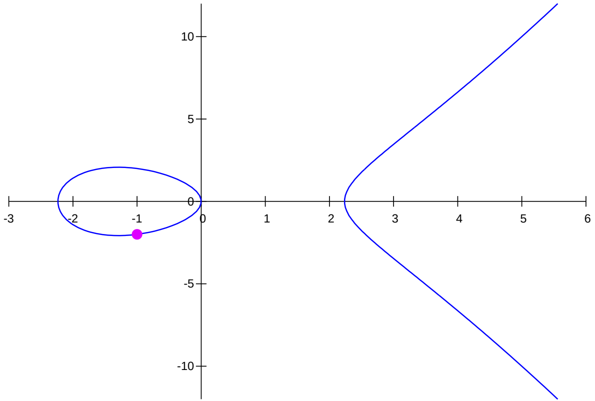
∞	$\longleftarrow$	$[0 : 1 : 0]$

An Elliptic Curve and a Point

$E : y^2 = x^3 - 5x$ over $\mathbb{Q}$, $\qquad p = (-1, -2) \in E(\mathbb{Q})$



Point Arithmetic — Cord & Tangent Law

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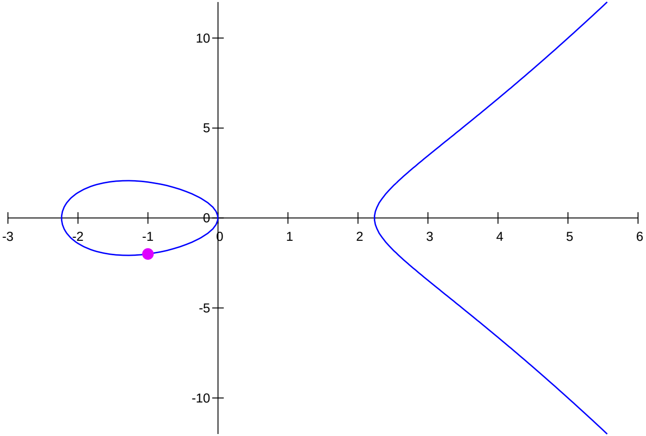
  - Must then invert $-r$ to obtain r.

Inverses on Elliptic Curves



$$-(\bullet) = ?$$

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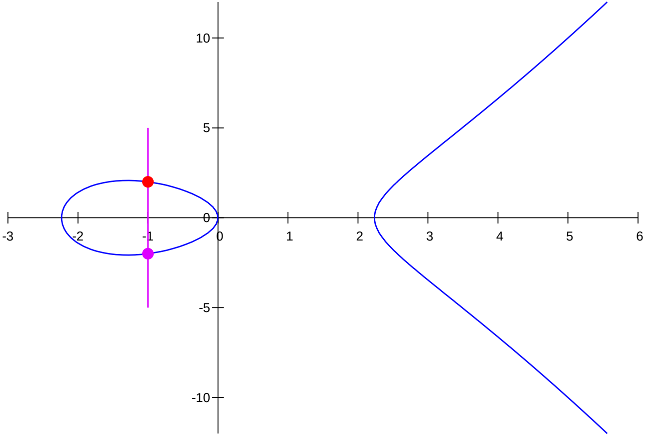

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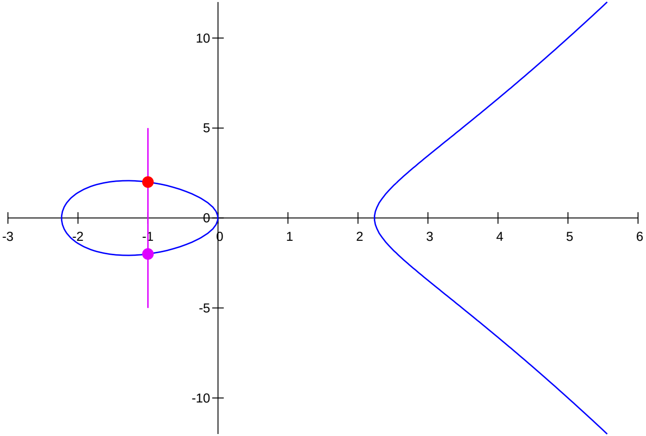
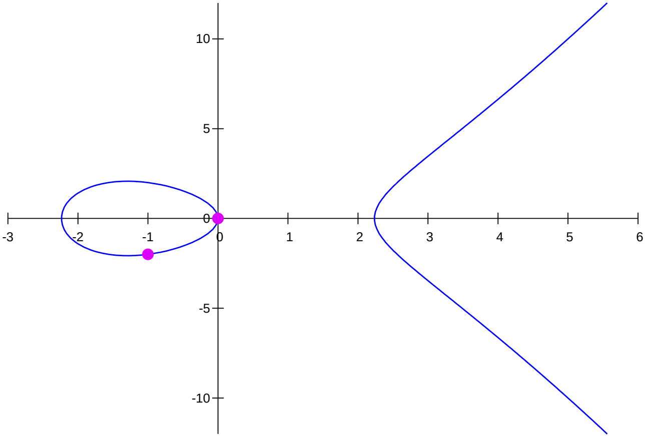
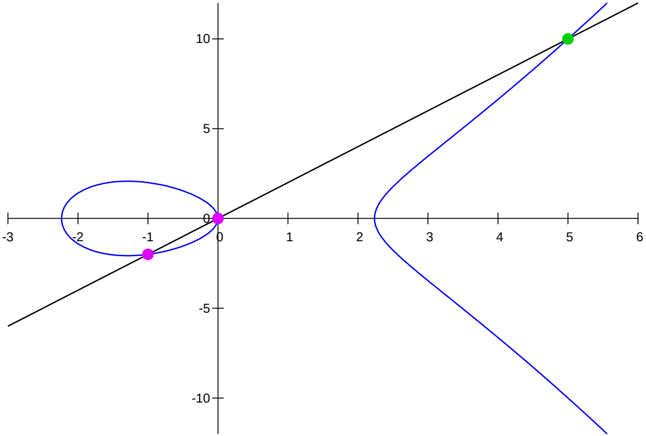


$$\color{magenta}{\bullet} + \color{red}{\bullet} + \infty = 0 \qquad \Rightarrow \qquad -(\color{magenta}{\bullet}) = \color{red}{\bullet}$$


10
5
0
-5
-10
-3
-2
-1
0
1
2
3
4
5
6

● + ● = ?

Addition on Elliptic Curves



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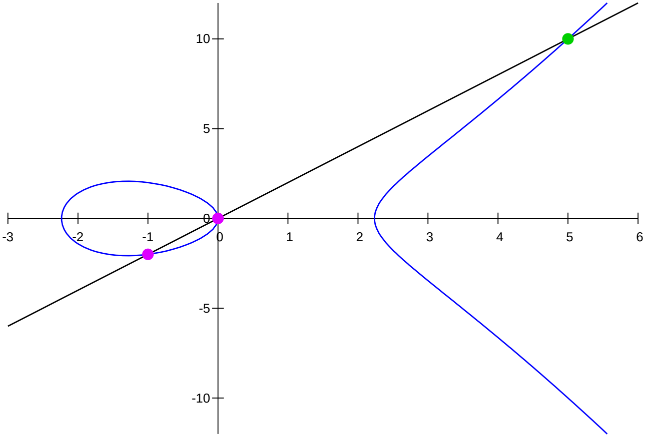

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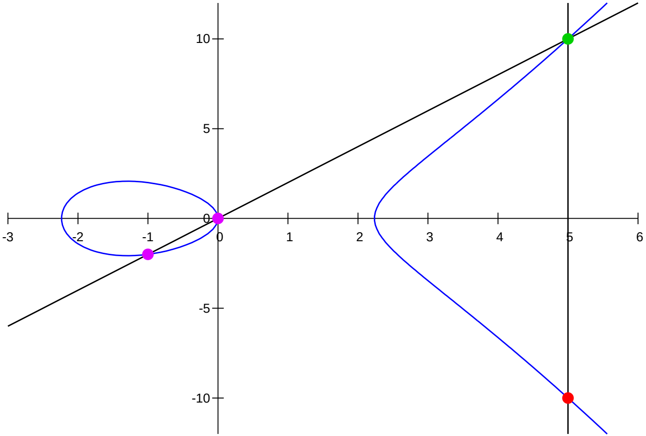

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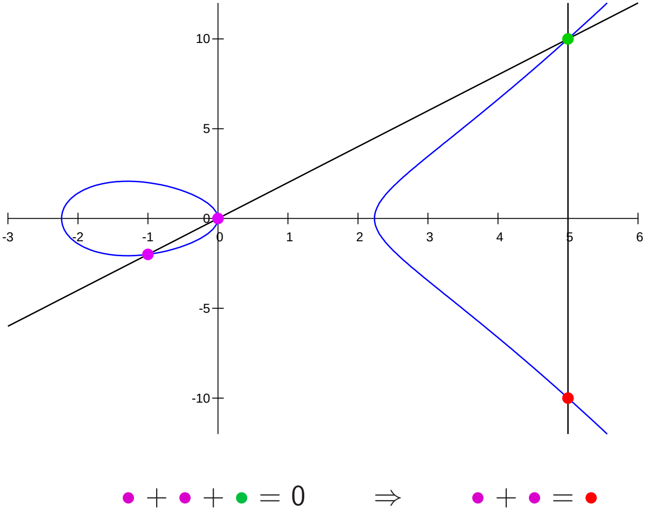
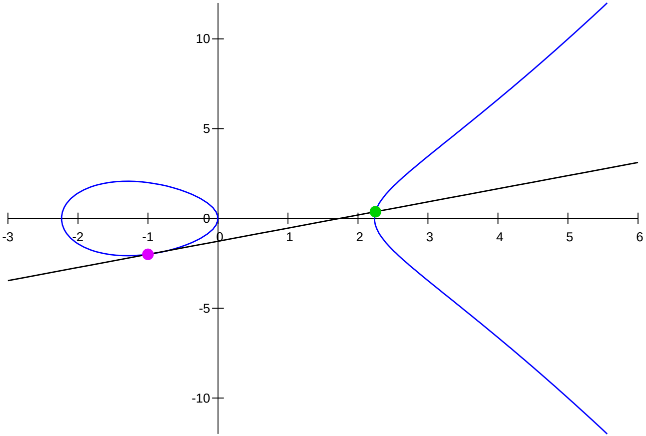


Doubling on Elliptic Curves



$$2 \times \bullet = ?$$

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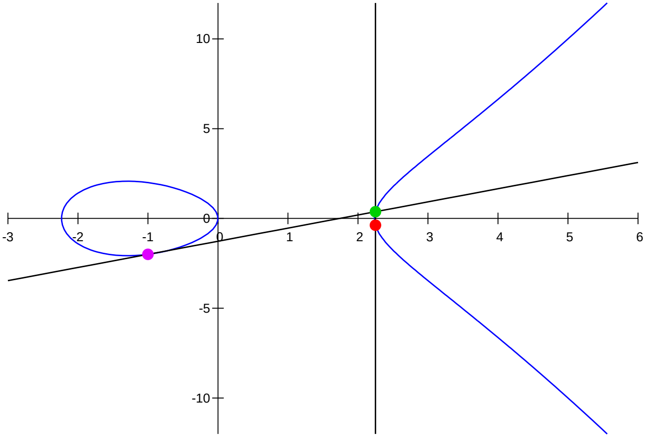

$$2 \times \bullet + \bullet = 0 \qquad \Rightarrow \qquad 2 \times \bullet = \bullet$$

Let

$$P_1 = (x_1, y_1),\ P_2 = (x_2, y_2) \qquad (P_1 \neq \infty,\ P_2 \neq \infty,\ P_1 + P_2 \neq \infty)\ .$$

Then

$$\begin{aligned} -P_1 &= (x_1, -y_1) \\ P_1 + P_2 &= (\lambda^2 - x_1 - x_2,\ -\lambda^3 + \lambda(x_1 + x_2) - \mu) \end{aligned}$$

where

$$\boxed{\lambda = \begin{cases} \dfrac{y_2 - y_1}{x_2 - x_1} & \text{if } P_1 \neq P_2 \\[2ex] \dfrac{3x_1^2 + A}{2y_1} & \text{if } P_1 = P_2 \end{cases}} \qquad \boxed{\mu = \begin{cases} \dfrac{y_1 x_2 - y_2 x_1}{x_2 - x_1} & \text{if } P_1 \neq P_2 \\[2ex] \dfrac{-x_1^3 + Ax_1 + 2B}{2y_1} & \text{if } P_1 = P_2 \end{cases}}$$

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- *Let* $(x_0, y_0) \in E(K) \setminus \{\infty\}$. *Then exists a unique* $P_{(x_0,y_0)} \in \mathbb{P}_1(F)$ *such that* $\mathrm{supp}(\mathrm{div}(x - x_0)) \cap \mathrm{supp}(\mathrm{div}(y - y_0)) = \{P_{(x_0,y_0)}, P_\infty\}$.

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- *The map* $\Psi : E(K) \to \mathbb{P}_1(K)$ *via* $(x_0, y_0) \mapsto P_{(x_0,y_0)}$ *and* $\infty \mapsto P_\infty$ *is a group isomorphism.*

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So we have group isomorphisms

$$(E(K), \text{point addition}) \overset{\Psi}{\longleftrightarrow} (\mathbb{P}_1(F), \oplus) \overset{\Phi}{\longleftrightarrow} (\mathrm{Cl}^0(F), \text{divisor addition})$$

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Description: Write $F = K(x, y)$ with $[F : K(x)] = 2$.
Then $F/K(x)$ has a minimal polynomial of the form

$$y^2 + h(x)y = f(x)$$

where $h(x)$ and $f(x)$ are polynomials (after we make everything integral) and $h(x) = 0$ if K has characteristic $\neq 2$.

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Remark: The case $g = 1$ and $\deg(f)$ odd also covers elliptic curves.

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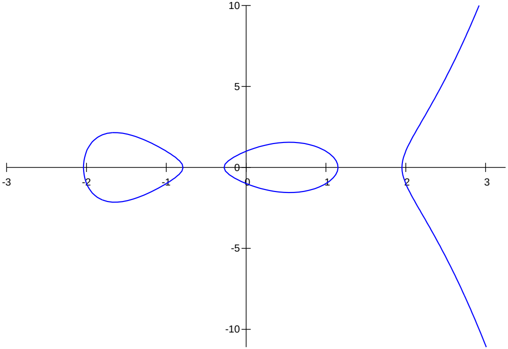

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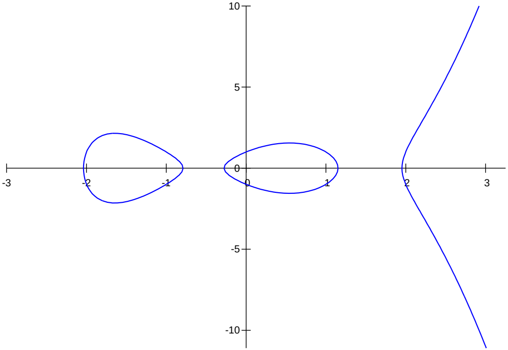

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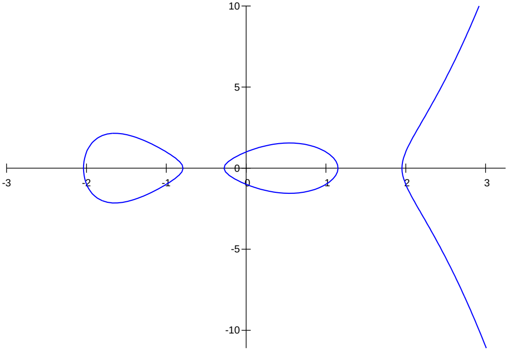

Note that the cord & tangent law no longer works when $g \geq 2$. In fact, any injection $\Phi : \mathbb{P}_1(F) \to Cl^0(F)$ is no longer surjective.

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Then the infinite place of $K(x)$ splits in F.

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Let $\mathrm{sgn}(f)$ denote the leading coefficient of $f(x)$.

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Theorem

- *Suppose $F/K(x)$ is ramified, with infinite place $P_\infty \in \mathbb{P}(F)$. Then every degree divisor class in $Cl^0(F)$ contains a unique divisor of the form*

$$D = D_0 - \deg(D_0) P_\infty \ ,$$

where D_0 is effective, $\deg(D_0) \le g$ and $P'_\infty \notin supp(D_0)$.

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where D_0 is effective, $\deg(D_0) \leq g$, $P_{\infty,1}, P_{\infty,2} \notin supp(D_0)$ and $-\lceil g/2 \rceil \leq n \leq \lfloor g/2 \rfloor - \deg(D_0)$.

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The divisor D is said to be reduced.

Remarks:

- D is uniquely determined by D_0 when $F/K(x)$ is ramified and by the pair (D_0, n) when $F/K(x)$ is split.
- "Generically" (i.e. for almost all classes in $\mathrm{Cl}(F)$), unless K is small, we have $\deg(D_0) = g$ and hence

  $D = D_0 - gP_\infty$ when $F/K(x)$ is ramified;

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Question: How to efficiently compute the reduced divisor in $[D_1 + D_2]$?

Let $(x_0, y_0) \in K \times K$ be a rational point on C, i.e.

$$y_0^2 + h(x_0)y_0 = g(x_0) \ .$$

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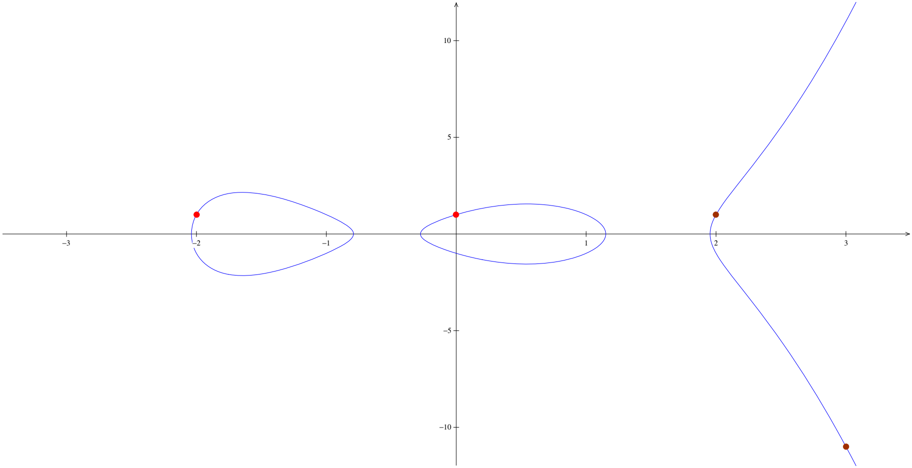
A divisor of the form

$$D = \sum_{i=1}^{r} P_i \in \mathrm{Div}(F) \ \text{ with } \ P_i \in \mathbb{P}_1(F) \ \text{ for all } i$$

can thus be identified with a multiset of r rational points on C.

Example, Genus 2, Ramified Model

$D_1 = P_{(-2,1)} + P_{(0,1)}$, $D_2 = P_{(2,1)} + P_{(3,-11)}$



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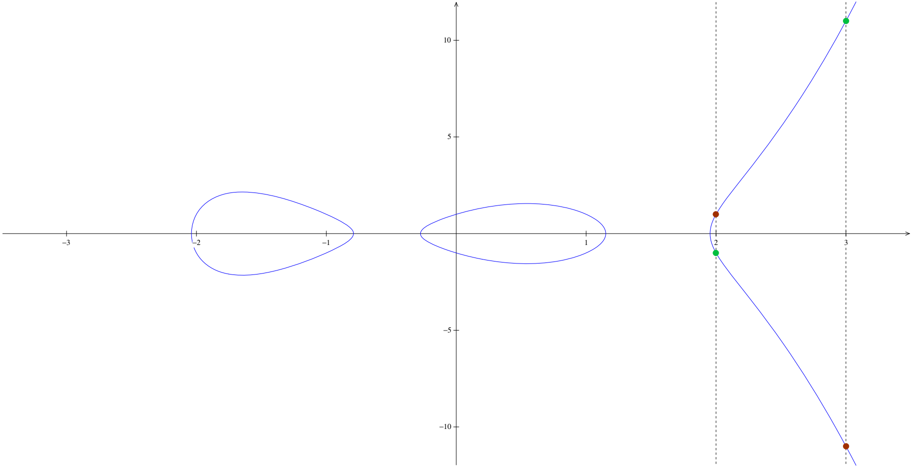
Degree 2 divisor class addition:

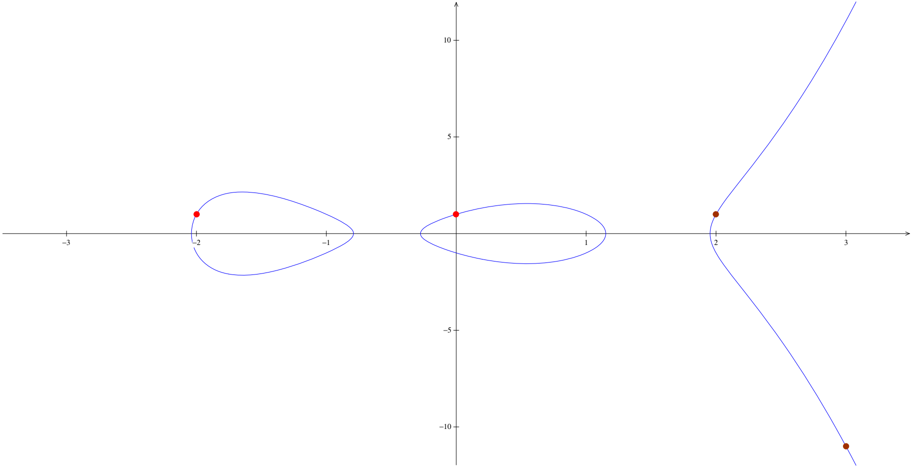
- Identity: $[0]$ ($D_0) = 0$).
- Inverses: invert points as before; the inverse of a divisor D consists of the inverses of the points in $\mathrm{supp}(D)$.
- Addition: "Any three degree 2 divisors on C lying on a cubic sum to zero."

Inverses in Genus 2, Ramified Models



$-(\bullet + \bullet) = \bullet + \bullet$

Addition in Genus 2, Ramified Models



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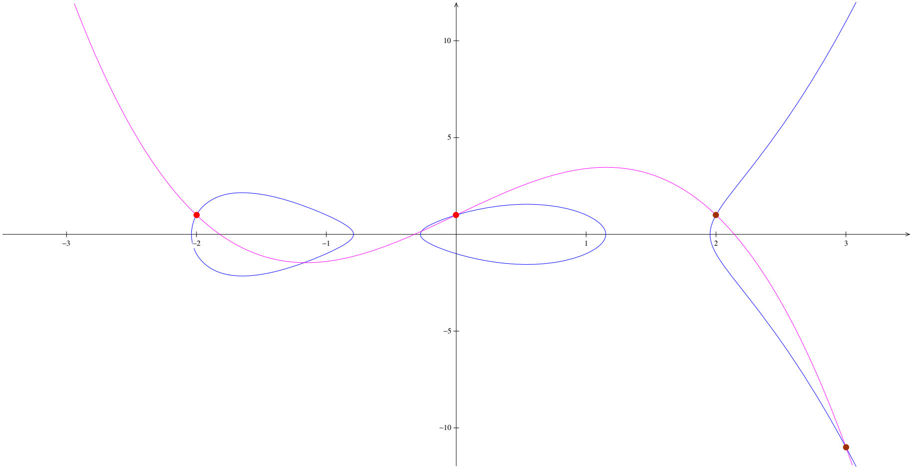

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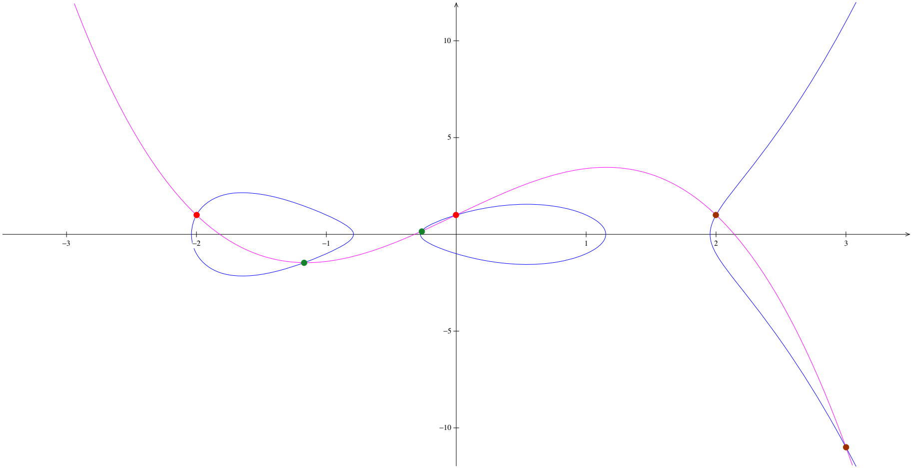

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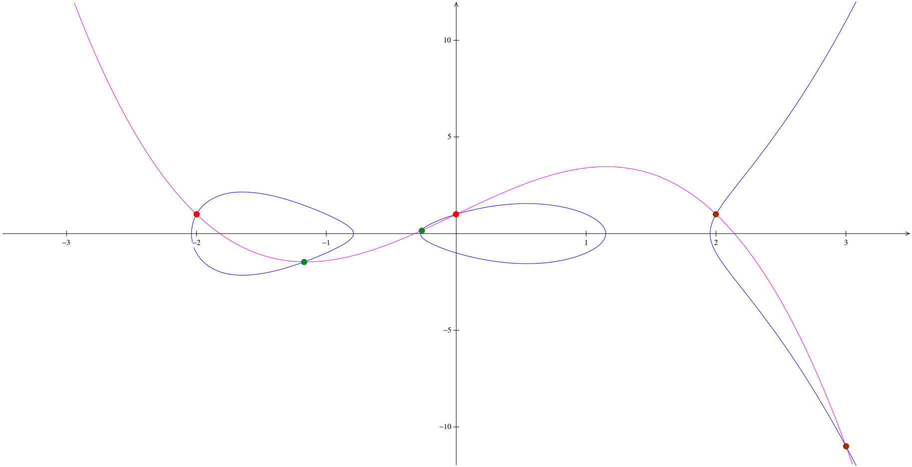

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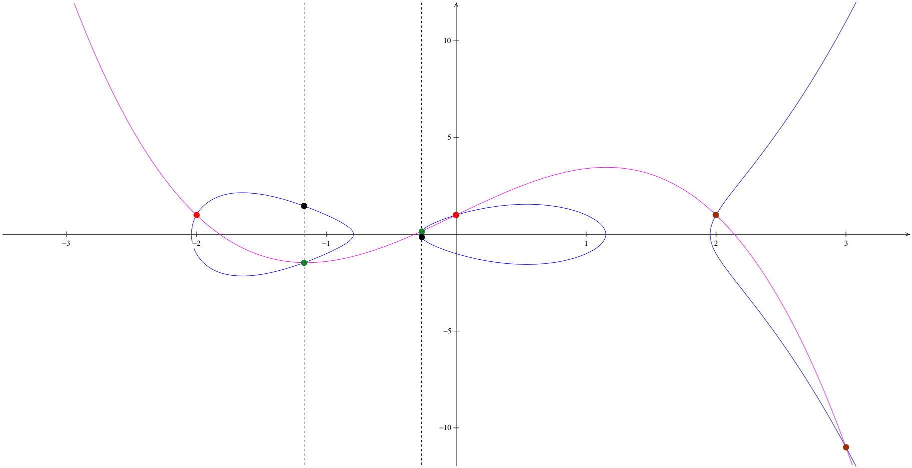

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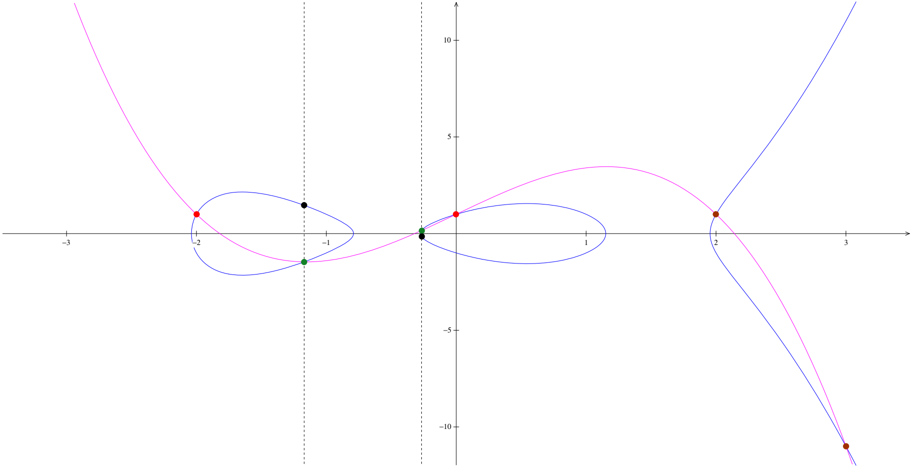

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- $[P_1 + P_2 - 2P_\infty] + [Q_1 + Q_2 - 2P_\infty] + [-R_1 - R_2 - 2P_\infty] = [0]$.
- So $[P_1 + P_2 - 2P_\infty] + [Q_1 + Q_2 - 2P_\infty] = [R_1 + R_2 - 2P_\infty]$.

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Consider $C : y^2 = f(x)$ with $f(x) = x^5 - 5x^3 + 4x + 1$ over $\mathbb{Q}$.

To add $[P_{(-2,1)} + P_{(0,1)} - 2P_\infty]$ and $[P_{(2,1)} + P_{(3,-11)} - 2P_\infty]$:

- The unique cubic through $(-2,1)$, $(0,1)$, $(2,1)$ and $(3,-11)$ is $y = v(x)$ with $v(x) = -(4/5)x^3 + (16/5)x + 1$.
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Remark: $u(x), v(x)$ have coefficients in K.

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5. Output $D_1 + D_2 = [u, v]$.

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General Arithmetic on Ramified Models

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- Mumford representations $[u, v]$ uniquely determine a reduced divisor and satisfy
$$\deg(v) < \deg(u) \leq g \ .$$
- Identity and Inverses as before.
- Addition Motto: "Any three divisors on C lying on a function of degree $\leq 2g - 1$ sum to zero."

Addition on Genus g Ramified Models

Let $D_1 = P_1 + \cdots + P_r$ and $D_2 = Q_1 + \cdots + Q_s$ ($r, s \leq g$) be disjoint.

*If $\deg(D) = g + 1$ in the last iteration, then the equation has $2g + 1$ roots.
In this case, $\deg(D)$ decreases by 1 only, from $g + 1$ to g.

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Let $D_1 = P_1 + \cdots + P_r$ and $D_2 = Q_1 + \cdots + Q_s$ $(r, s \leq g)$ be disjoint.

To add $[D_1 - rP_\infty]$ and $[D_2 - sP_\infty]$:

1. Put $D = P_1 + \cdots + P_r + Q_1 + \cdots + Q_s \quad // \; (\deg(D) = r + s \leq 2g)$.
2. Repeat until $\deg(D) \leq g$ (up to $\lceil g/2 \rceil$ times):
    1. Compute the unique function $y = v(x)$ with $\deg(v) = \deg(D) - 1$ through the points in $\operatorname{supp}(D)$.
    2. The equation $v^2 + hv - f = 0$ has $2\deg(D) - 2$ roots.*
       $\deg(D)$ of these are the x-coordinates of the points in $\operatorname{supp}(D)$.
       Denote the remaining roots by $x_1, \ldots, x_{\deg(D)-2}$.
    3. Substitute the x_i into $y = v(x)$, i.e. compute $y_i = v(x_i)$ and put $-R_i = P_{(x_i, y_i)}$, for $1 \leq i \leq \deg(D) - 2$.
    4. Put $D = R_1 + R_2 + \cdots + R_{|D|-2}$.
3. Output $[D - \deg(D)P_\infty]$.

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Example: if $D = P_{(x_0, y_0)}$ (a prime divisor), then $u(x) = x - x_0$, $v(x) = y_0$.

Addition Via Mumford Representations

Let $D_1 = [u_1, v_1]$, $D_2 = [u_2, v_2]$ be disjoint divisors.

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Final Remarks

Adding non-disjoint divisors via their Mumford representation is slightly more complicated, but can also be done with a simple polynomial arithmetic and two gcd calculations.

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However, unless K is small, we know that $n = -\lceil g/2 \rceil$ almost certainly, so there is no need.

References

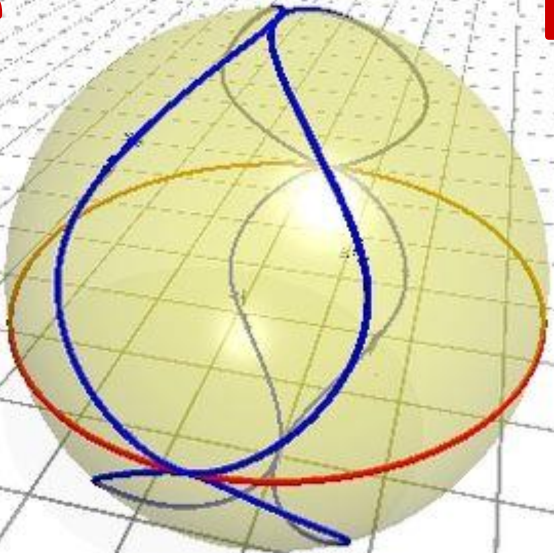
For divisor class arithmetic on ramified models:

- Alfred J. Menezes, Yi-Hong Wu and Robert J. Zuccherato, *An elementary introduction to hyperelliptic curves*, CORR 96-19, University of Waterloo 1996; Also appeared as an appendix in: Neal Koblitz, *Algebraic Aspects of Cryptography*, Algorithms and Computation in Mathematics, vol. 3. Springer, Berlin, 1998.

For divisor class arithmetic on split models:

- Steven D. Galbraith, Michael Harrison and David J. Mireles Morales, Efficient hyperelliptic arithmetic using balanced representation for divisors. In *Algorithmic Number Theory*, Lecture Notes in Computer Science, vol. 5011, Springer, Berlin, 2008, 342–356.

The End



$$y^2 = x^6 + x^2 + x$$

http://voltage.typepad.com/superconductor/2011/09/a-projective-imaginary-hyperelliptic-curve.html