Categories

Robin Cockett

Department of Computer Science
University of Calgary
Alberta, Canada

robin@cpsc.ucalgary.ca

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DEFINITION

A **category**, $\mathbb{C}$ consists of a *directed graph*:

- A collection of **objects**, $\mathbb{C}_0$.
- A collection of **maps**, $\mathbb{C}_1$.
- Each map $f \in \mathbb{C}_1$ has a **domain** $\partial_0(f) \in \mathbb{C}_0$ and a **codomain** $\partial_1(f) \in \mathbb{C}_0$.

  A map f with domain A and codomain B is written
  $$f : A \longrightarrow B \text{ or } A \xrightarrow{f} B$$

DEFINITION cont.
with a *composition*:

- ▶ Associated with each object is an identity map:
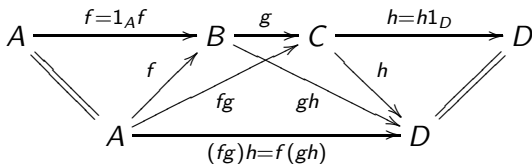
$$\overline{1_A : A \longrightarrow A}$$

- ▶ Any pair of maps $f : A \longrightarrow B$ and $g : B \longrightarrow C$ (with the codomain of f being the same as the domain of g) can be **composed**[1] to obtain $fg : A \longrightarrow C$:

$$\frac{f : A \longrightarrow B \quad g : B \longrightarrow C}{fg : A \longrightarrow C}$$

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[1]NOTE: I use diagrammatic order for composition.

DEFINITION cont. Such that:

- (Identity laws) if $f : A \longrightarrow B$ then $1_A f = f = f 1_B$,
- (Associative law) if $f : A \longrightarrow B$, $g : B \longrightarrow C$, and $h : C \longrightarrow D$ then $(fg)h = f(gh)$.

MOTIVATING EXAMPLE I

Sets and functions, Set:

Objects: Sets.

Maps: $f : A \longrightarrow B$ is a function. That is a relation $f \subseteq A \times B$ which is **deterministic** ($a\ f\ b_1$ and $a\ f\ b_2$ implies $b_1 = b_2$) and **total** ($\forall a \in A.\exists b \in B.a\ f\ b$);

Identities: $1_A : A \longrightarrow A$ is the diagonal relation $\Delta_A = \{(a, a) | a \in A\} \subseteq A \times A$;

Composition: Relational composition $fg = \{(a, c) | \exists b.(a, b) \in f\ \&\ (b, c) \in g\}$;

Composition is associative and the identities behave correctly!

MOTIVATING EXAMPLE II

Sets and *partial* functions, Par:

Objects: Sets.

Maps: $f : A \longrightarrow B$ is a *partial* function. That is a relation
$f \subseteq A \times B$ which is **deterministic** ($a\ f\ b_1$ and $a\ f\ b_2$
implies $b_1 = b_2$);

Identities: $1_A : A \longrightarrow A$ is the diagonal relation
$\Delta_A = \{(a, a) | a \in A\} \subseteq A \times A$;

Composition: Relational composition
$fg = \{(a, c) | \exists b.(a, b) \in f\ \&\ (b, c) \in g\}$;

Composition is associative and the identities behave correctly!

A category is not determined by its objects!

SUBCATEGORIES
A **subcategory** of a category is determined by taking a subset of the objects and the maps which form a category.

A subcategory is a **full subcategory** if $f : A \longrightarrow B$ is in the category whenever the objects A and B are in the subcategory. Clearly full subcategories are completely determined by the objects which are included.

Set is a subcategory of Par BUT it is not a full subcategory.

MOTIVATING EXAMPLE III

Sets and relations, Rel:

Objects: Sets.

Maps: $f : A \longrightarrow B$ is a relation $f \subseteq A \times B$;

Identities: $1_A : A \longrightarrow A$ is the diagonal relation
$\Delta_A = \{(a, a) | a \in A\} \subseteq A \times A$;

Composition: Relational composition
$fg = \{(a, c) | \exists b.(a, b) \in f \ \& \ (b, c) \in g\}$;

Composition is associative and the identities behave correctly!

This category has a *converse* operation

$$\frac{f : A \longrightarrow B}{f^\circ : B \longrightarrow A}$$

where $(1_A)^\circ = 1_A$, $(f^\circ)^\circ = f$, and $(fg)^\circ = g^\circ f^\circ$.

MOTIVATING EXAMPLE IV

Sets and relations, Rel':

Objects: Sets.

Maps: $f : A \longrightarrow B$ is a relation $f \subseteq A \times B$;

Identities: $1_A : A \longrightarrow A$ is the off-diagonal relation
$\Delta_A = \{(a, a') | a, a' \in A, a \neq a'\} \subseteq A \times A$;

Composition: Dual relational composition
$fg = \{(a, c) | \forall b.(a, b) \in f \ \lor \ (b, c) \in g\}$;

Composition is associative and the identities behave correctly!

A category is not determined by its sets and maps!

LARGE AND SMALL
These are important examples ...

Note the "set of sets" is not a set so the objects of these categories do not form a set! These are examples of "large" categories.

If $\mathbb{X}$ is a category then the **homset** $\mathbb{X}(A, B)$ consists of all the arrows $f : A \longrightarrow B$.

Notice that in all these examples the homsets do form sets! When the homsets live in sets we say the category is **locally small**. When the objects are a set as well we say the category is **small**

In fact, more generally the homsets can live in *another category* and when this happens we say the category is **enriched** in that other category. So small categories are enriched in Set!

OTHER EXAMPLES

For any algebraic theory algebras and homomorphisms form a category:

- ► The category of groups Group: object groups, maps group homomorphisms (preserve composition and unit).
- ► The category of meet semilattices MeetSLat: objects meet semilattices, maps semilattice homomorphism (preserve meet, and top).
- ► The category of commutative rings, CRing: objects commutative (unital) rings, ring homomorphisms (preserve addition, multiplications. and both units)

These are all large categories which are locally small.

MOTIVATING EXAMPLE V

Finite sets and functions, Set_f:

Objects: Finite sets.

Maps: $f : A \longrightarrow B$ is a function. That is a relation
$f \subseteq A \times B$ which is **deterministic** ($a \ f \ b_1$ and $a \ f \ b_2$
implies $b_1 = b_2$) and **total** ($\forall a \in A . \exists b \in B . a \ f \ b$;

Identities: $1_A : A \longrightarrow A$ is the diagonal relation
$\Delta_A = \{(a, a) | a \in A\} \subseteq A \times A$;

Composition: Relational composition
$fg = \{(a, c) | \exists b . (a, b) \in f \ \& \ (b, c) \in g\}$;

Clearly this category is enriched in finite sets! (Although its objects do not form a finite set ...)

It is also a *full* subcategory of Set.
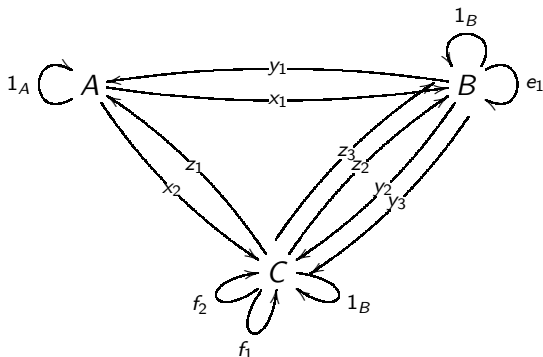
MOTIVATING EXAMPLE VI
A category can be finite.

The simplest category of all **0** has no objects and no maps! This is called (for reasons which will be explained later) the **initial** category. The initial category is certainly finite and there is not much else one can say about it!

The next most simple category is the category with exactly one object and exactly one arrow, **1**. This is called the **final category**: it is also finite and there is not so very much more one can say about it either! The one arrow is actually forced to be the identity map on the one object.

MOTIVATING EXAMPLE VI cont.

A finite category, a category *internal* to finite sets, must have both
a finite number of objects and a finite number of arrows.

A finite category, $\mathbb{F}$, may be presented as a directed graph with a
multiplication table for each object:

MOTIVATING EXAMPLE VI cont.

		A	B	C
	A	1_A	x_1	x_2
A	1_A	1_A	x_1	x_2
B	y_1	y_1	e_1	y_3
C	z_1	z_1	z_3	f_2

		A	B		C	
	B	y_1	1_B	e_1	y_2	y_3
A	x_1	1_A	x_1	x_1	x_2	x_2
B	1_B	y_1	1_B	e_1	y_2	y_3
	e_1	y_1	e_1	e_1	y_3	y_3
C	z_2	z_1	z_2	z_3	f_1	f_2
	z_3	z_1	z_3	z_3	f_2	f_2

		A	B		C		
	C	z_1	z_2	z_3	1_C	f_1	f_2
A	x_2	1_A	x_1	x_1	x_2	x_2	x_2
B	y_2	y_1	1_B	e_1	y_2	y_2	y_3
	y_3	y_1	e_1	e_1	y_3	y_3	y_3
C	1_C	z_1	z_2	z_3	1_C	f_1	f_2
	f_1	z_1	z_2	z_3	f_1	f_1	f_2
	f_2	z_1	z_3	z_3	f_2	f_2	f_2

Composition tables for $\mathbb{F}$

FINITE CATEGORIES ...

Finite categories are very important ... they are great for providing simple counterexamples.

Can you find categories with:

		0	1	2	3
	0	1	0	0	0
	1	0	1	0	0
Number of maps	2	0	2	1	0
	3	0		3	1
	4	0			

Number of objects

Categories with only one object are **monoids**.

DUALITY ..
Category theory is full of symmetries ...

The basic source of symmetry is the ability to reverse arrows.
Given any category we may obtain a new category by keeping
everything the same except to switch the direction of the arrows. If
we start with a category $\mathbb{C}$ and flip the direction of the arrows we
obtain a new category written $\mathbb{C}^{\mathrm{op}}$ (the **dual category**). Observe
now that anything which is true of $\mathbb{C}$ now holds in the **dual** form
in $\mathbb{C}^{\mathrm{op}}$.

Thus, when we prove a result there is always another result,
obtained by reversing the sense, of the arrows which will also be
true. This principle of duality allows us to get double the bang for
our buck!

DUALITY ..

- What is the category $\mathrm{Rel}^{\mathrm{op}}$?
- What is the category $\mathbb{F}^{\mathrm{op}}$?
- What is the category $\mathrm{Set}^{\mathrm{op}}$?
- What is the category $\mathrm{CRing}^{\mathrm{op}}$?

Obvious questions don't always have easy answers!

PATHS ... EXAMPLE VII

Given a directed graph $\mathbb{G}$ we may form a category Path($\mathbb{G}$), called the **path** category of $\mathbb{G}$:

Objects: The objects are the nodes of $\mathbb{G}$.

Maps: Sequences of edges in $\mathbb{G}$: $(A, [g_0, .., g_n], B) : A \longrightarrow B$
where, when the list of maps is non-empty,
$A = D_0(g_0)$, $B = D_1(g_n)$, and $D_1(g_i) = D_0(g_{i+1})$.
Otherwise $A = B$.

Identities: $(A, [], A)$ for each object A.

Composition: $(A, l_1, B)(B, l_2, C) = (A, l_1 \mathbin{++} l_2, C)$.

MATRICES ... EXAMPLE VIII

Let R be a **rig**: a commutative associate operation, addition $(x + y$ with an identity $0)$ and an associative multiplication with unit 1 such that:

$$x \cdot (y + z) = x \cdot y + x \cdot z \quad \text{and} \quad (y + z) \cdot x = y \cdot x + z \cdot x,$$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \quad \text{and} \quad x \cdot 1 = x = 1 \cdot x$$

$$x \cdot 0 = 0 = 0 \cdot y$$

then we may form $\text{Mat}(R)$ the category of R–matrices:

   Objects: The natural numbers $0, 1, 2, ...$

      Maps: $n \times m$-matrices $[r_{i,j}] : n \longrightarrow m$

   Identities: $1_n : n \longrightarrow n$ the diagonal matrix.

Composition: Matrix multiplication.

We allow $0 \times n$ and $n \times 0$ matrices! The composites with these "empty" matrices are themselves empty.

MATRICES ... EXAMPLE VIII cont.

Mat(R) always has a converse involution given by transposition.

A special example of this category which is very well-studied is Mat(K) where K is a field (such as $\mathbb{R}$). The category is then equivalent to the category of finite dimensional vector spaces.

Mat(R) is a small category.

What is Mat(Bool) where Bool is a rig with the meet giving the multiplication and the join the addition.

MONICS, EPICS

A map $f : A \longrightarrow B$ in a category $\mathbb{C}$ is **monic** in case whenever $k_1 f = k_2 f$ then $k_1 = k_2$.

Dual: a map $f : A \longrightarrow B$ in a category $\mathbb{C}$ is **epic** (or sometimes an epimorphism) in case whenever $f h_1 = f h_2$ then $h_1 = h_2$.

The fact that a map is monic does not stop it from being epic as well: a map that is both epic and monic we shall refer to as being **bijic**.

In Set monic=injective, epic=surjective ...
What are the epics in Path($\mathbb{G}$).

MONICS, EPICS

Lemma
In any category $\mathbb{C}$:

(i) *The composition of monics is monic;*

(ii) *Dual: the composition of epics is epic;*

(iii) *If fg is monic then f is monic;*

(iv) *Dual: if fg is epic then g is epic.*

(v) *The composition of bijics is bijic;*

(vi) *If fg is bijic then f is monic and g is epic.*

SECTIONS, RETRACTIONS, ISOMORPHISMS

A map $f : A \longrightarrow B$ is a **section** in case there is a map $f' : B \longrightarrow A$ such that $ff' = 1_A$.

Dual: a map $g : A \longrightarrow B$ is a **retraction** in case there is a map $g' : B \longrightarrow A$ such that $g'g = 1_B$. It is quite possible for a map to be both a section and a retraction: such a map is called an **isomorphism**. Clearly identity maps are always isomorphisms.

SECTIONS, RETRACTIONS, ISOMORPHISMS

Lemma

In any category $\mathbb{C}$:

(i) *All sections are monic;*

(ii) *Dual: all retractions are epic;*

(iii) *The composition of two sections is a section;*

(iv) *Dual: the composition of two retractions is a retraction;*

(v) *If fg is a section then f is a section;*

(vi) *Dual: if fg is a retraction then g is a retraction;*

(vii) *All isomorphisms are bijic;*

(viii) *Composites of isomorphisms are isomorphisms;*

(ix) *If fg is an isomorphism then f is a section and g is a retraction.*

ISOMORPHISMS

If $f : A \longrightarrow B$ is a map a **right inverse** for f is a map $g : B \longrightarrow A$ such that $fg = 1_A$. Dually **left inverse** for f a map $h : B \longrightarrow A$ such that $hf = 1_B$. Right inverses are sections, left inverses are retractions. An isomorphism has both a left inverse and a right inverse:

Lemma

If $f : A \longrightarrow B$ has a left inverse h and a right inverse g then $h = g$.

PROOF: Observe f is both epic and monic as it is both a section and a retraction. Thus, $fh = fh1_A = fhfg = f1_Bg = fg = 1_A$ so that h is also a right inverse of f. But then $fh = fg$ and as f is epic $h = g$. $\qquad\square$

Thus the inverse of an isomorphism, f, is unique we shall denote it f^{-1}.

ISOMORPHISMS

We have the following alternative characterizations of isomorphisms:

Lemma

The following are equivalent:

(i) *f is a section and a retraction;*   (iii) *f is an epic section;*
(ii) *f is a monic retraction*   (iv) *f is an isomorphism;*

$gf = 1$ implies $fgf = f$ as f is monic $fg = 1$.

When a category has *every* map an isomorphism it is called a **groupoid**. All groupoids have a converse:

$$(\_)^{-1} : \mathbb{G} \longrightarrow \mathbb{G}; f \mapsto f^{-1}$$

The isomorphism of any category form a groupoid.

PREORDERS ... EXAMPLE IX
We can enrich in sets with cardinality at most one! The result
preorders!

When there is at most one arrow between any two objects the
value of the composite of any two maps is forced! Thus, it is
simply a matter of whether maps exists between objects or not.

A (small) **preorder** is a set with a reflexive, transitive relation . A
relation is **reflexive** on a set X in case whenever $x \in X$ we require
(x, x) to be in the relation: this gives the identity map on that
object. A relation is **transitive** in case whenever (x, y) and (y, z)
are in the relation then (x, z) must be in the relation: this gives
composition.

A relation is an **equivalence relation** in case in addition it is
symmetric that is whenever (x, y) is in the relation (y, x) is also
in the relation. This is equivalent to asking that every arrow is an
isomorphism.

A **partially order set** is a preorder with the addition
anti-symmetry property that whenever (x, y) and (y, x) are in the
relation then $x = y$. This is equivalent categorically to asking
either that the only isomorphisms are the identity maps.

MONICS, EPICS

All maps in preorders are bijic ...

What are the monics in Par?

What are the monics in Rel?

Simple questions don't necessarily have simple answers!!

PRODUCTS AND SUMS OF CATEGORIES

Given two categories $\mathbb{X}$ and $\mathbb{Y}$ one can just consider the disjoint union of the categories $\mathbb{X} \sqcup \mathbb{Y}$. This is a category whose objects (respectively maps) are the disjoint union of the objects (respectively maps) of each $\mathbb{X}_i$.

Clearly objects from different components will not be connected by any maps. Indeed given any category there is always a (unique) decomposition of it into connected components which allows us to view it as a sum of connected categories.

The product of two categories $\mathbb{X}$ and $\mathbb{Y}$, $\mathbb{X} \times \mathbb{Y}$, puts the categories in parallel:

   Objects: Pairs (X, Y) with $X \in \mathbb{X}$ and $Y \in \mathbb{Y}$.

   Maps: $(f, g) : (X, Y) \longrightarrow (X', Y')$ where $f : X \longrightarrow X'$ in $\mathbb{X}$ and $g : Y \longrightarrow Y'$ in $\mathbb{Y}$.

   Identities: $(1_X, 1_Y) : (X, Y) \longrightarrow (X, Y)$.

Composition: $(f, g)(f', g') = (ff', gg')$.
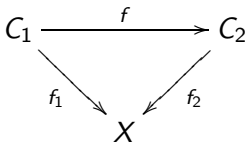
SLICE CATEGORIES
Turning maps into objects!!!

If $\mathbb{C}$ is any category and $X \in \mathbb{C}$ we may form the **slice category** $\mathbb{C}/X$. This has the following structure:

Objects: Maps of $\mathbb{C}$ to X, $f : C \longrightarrow X$;

Maps: Triples $(f_1, g, f_2) : f_1 \longrightarrow f_2$ which are commutative triangles:

$$C_1 \xrightarrow{\ \ f\ \ } C_2$$
$$\underset{f_1}{\searrow} \quad \underset{f_2}{\swarrow}$$
$$X$$

Identities: $(f, 1_C, f) : f \longrightarrow f$.

Composition: $(f_1, g, f_2)(f_2, h, f_3) = (f_1, gh, f_3)$ which is well-defined as $ghf_3 = gf_2 = f_1$.

What is Set/A? – A-indexed sets.

IDEMPOTENTS

An endomap $e : A \longrightarrow A$ is an **idempotent** if $ee = e$.

If $h : A \longrightarrow B$ is a retraction with left inverse $h' : B \longrightarrow A$ then hh' is an idempotent as $hh'hh' = h1_B h' = hh'$.

We shall say that an idempotent e is **split** if there is a retraction h with left inverse h' such that $e = hh'$. Splittings are *unique up to unique isomorphism*:

Lemma

Suppose $e : A \longrightarrow A$ is an idempotent and $h_1 : A \longrightarrow B_1$ has left inverse h_1' and $h_2 : A \longrightarrow B_2$ has left inverse h_2', with $e = h_1 h_1' = h_2 h_2'$ then there is a unique isomorphism $k : B_1 \longrightarrow B_2$ such that $h_1 k = h_2$ and $k h_2' = h_1'$.

PROOF: Set $k = h_1' h_2$ then k is an isomorphism. Suppose k' also satisfies $h_1 k' = h_2$ then $h_1 k' = h_1 k$ and as h_1 is a retraction and therefore epic it follows that $k = k'$. $\qquad\square$

IDEMPOTENTS

Idempotents will not generally split, however, there is an IMPORTANT construction which allows one to freely split idempotents.

Let $\mathbb{C}$ be any category, define $\text{Split}(\mathbb{C})$ be the following category:

> Objects: Idempotents e of $\mathbb{C}$;
>
> Maps: $(e_1, f, e_2) : e_1 \longrightarrow e_2$ where $e_1 : A \longrightarrow A$ and $e_2 : B \longrightarrow B$ is a map $f : A \longrightarrow B$ in $\mathbb{C}$ such that $e_1 f e_2 = f$;

Compositions As in $\mathbb{C}$ on the middle coordinate:
$$(e_1, f, e_2)(e_2, g; e_3) = (e_1, fg, e_3).$$

> Identities: the identity for an idempotent is that idempotent $(e, e, e) : e \longrightarrow e$.

... it is not hard to show that this is a category.

IDEMPOTENTS

What is interesting about this category is that all the idempotents in it split:

Theorem

Let $\mathbb{C}$ be any category then $\mathbb{C}$ is a full subcategory of Split($\mathbb{C}$) *and all idempotents split in* Split($\mathbb{C}$).

PROOF: We may regard $\mathbb{C}$ as a full subcategory of Split by letting the identity maps (which are certainly idempotent) represent the objects of $\mathbb{C}$ in Split($\mathbb{C}$).

Suppose $(e, k, e) : e \longrightarrow e$ is an idempotent in Split($\mathbb{C}$) then k is an idempotent in $\mathbb{C}$. But then we have maps $(e, k, k) : e \longrightarrow k$ and $(k, k, e) : k \longrightarrow e$ in Split($\mathbb{C}$) and it is easy to check that these provide a splitting for (e, k, e). $\qquad\square$

IDEMPOTENTS
Why is this construction interesting?

Consider the category of partial recursive functions on the natural numbers, Rec. Each enumerable set may be characterized by an idempotent which is the computation which returns the element unchanged when it is in the recursively enumerable set but simply does not terminate on elements outside.

In Split(Rec) there is an object or type for each enumerable set. Thus, this gives an example of how to construct from a unityped system a very rich type system.

EXAMPLE X – TANGLES

Do we have time to tango!

EXAMPLE X – TANGLES

There are non-bijic tangles, because there are nontrivial idempotents: if e is any nontrivial idempotent, then e is not monic, because $ee = 1e$ but $e \neq 1$.

An example of an idempotent is the following:

We can also add an equal number of cups and caps to the left and right, and extend the length of the diagonal line accordingly. For example:
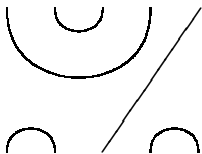

It is easy to see this is also an idempotent.

EXAMPLE X – TANGLES
However, the $1 \longrightarrow 1$ identity can be written in other ways. For example,

If we take the splitting corresponding to this writing of the identity, we recover another idempotent:

EXAMPLE X – TANGLES

Do idempotents split in tangles?