Categorical Aspects of Galois Theory

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Outline

Some Field Theory

First, let us recall a few notions from field theory...

- Let L/K be a field extension and $G \subset \text{Aut}_K(L)$. The **fixed field** of G is the field:

$$L^G = \{\alpha \in L \mid \forall g \in G, g\alpha = \alpha\}.$$

- A field extension L/K is called a **Galois extension** if L/K is algebraic and there exsts a subgroup $G \subset \text{Aut}_K(L)$ such that $K = L^G$.

The Galois Correspondence

For a Galois extension L/K define the **Galois group** to be $\mathrm{Gal}(L/K) := \mathrm{Aut}_K(L)$.

Theorem: Galois Correspondence

Let L/K be a Galois extension. Then we can define the following bijective correspondence:

$$\{\text{Intermediate fields } E \text{ of } L/K\} \longleftrightarrow \{\text{Subgroups } H \text{ of } Gal(L/K)\}$$
$$E \longmapsto Aut_E(L)$$
$$L^H \longleftarrow H$$

There are more Galois correspondences out there!

What is a Galois Category?

A category **C** together with a **fundamental functor** $F : \mathbf{C} \longrightarrow$ **fset** satisfying some conditions. . .

- (G1) **C** has a final object and pullbacks.
- (G2) **C** has an initial object, finite coproducts, and quotients by finite groups of automorphisms exist.
- (G3) Any $f : X \to Y$ in **C** can be written $f = e \circ m$, for e epic and m monic.
- (G4) F preserves final objects and pullbacks.
- (G5) F preserves initial objects, finite coproducts, epics, and quotients.
- (G6) For g a map in **C**, if $F(g)$ is an isomorphism, then g is an isomorphism.

Examples of Galois Categories

Galois Category **C**	Fundamental Functor F
fset - the category of finite sets.	$F = id_{\mathbf{fset}}$
π-**fset** - the category of finite sets with a continuous action by a profinite group π.	$F = \text{Forget} : \pi\text{-}\mathbf{fset} \longrightarrow \mathbf{fset}$
$\mathbf{SAlg}_K{}^{op}$ - the category of free separable K-algebras, for K a field.	$F(A) = \text{Hom}_{\text{Alg}_K}(A, K_s)$

Note: $K_s = \{x \in \overline{K} \mid x \text{ is separable over } K\}$.

Main Theorem of Galois Categories

Note: A **profinite group** is a topological group which is totally disconnected, compact, and Hausdorff.
Eg. Every finite group is profinite, given the discrete topology.

Theorem

Let **C** *be a small Galois category. Then there exists a profinite group* π *such that* **C** *is equivalent to* π*-**fset**.*

An Example of the Theorem in Practice

Consider $\mathbf{SAlg}_K{}^{op}$, the opposite of the category of free separable K-algebras, for K any field.
Recall: $K_s = \{x \in \overline{K} \mid x \text{ is separable over } K\}$. K_s is a Galois extension over K.

Proposition

There is an equivalence of categories:

$$\mathbf{SAlg}_K^{op} \simeq G - \mathbf{fset}$$

where G is the Galois group of K_s over K, i.e. $Gal(K_s/K)$.

The Family Category

Definition

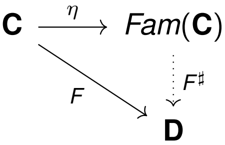
Let **C** be any category. Define the *family category*, denoted *Fam*(**C**), to be the category where:

objects: $(C_i)_{i \in I}$, indexed objects $C_i \in \mathbf{C}$.

maps: $(C_i)_{i \in I} \xrightarrow{(f,F)} (D_j)_{j \in J}$, where $f : I \longrightarrow J$ is a map in **set** and for each $i \in I$, $F(i) : C_i \longrightarrow D_{F(i)}$ is a map in **C**.

The Coproduct Completion

- We say that *Fam*(**C**) is the *coproduct completion* of **C**, which is described by the following universal property:



Where **D** is a category with coproducts, F is a functor, $\eta(C) = (C)_{\{*\}}$, i.e. η sends C to the singleton family, and $F^\sharp$ preserves coproducts.

Connected Objects

Definition

An object $X \in \mathbf{C}$ is called *connected* if:

$$Hom(X, -) : \mathbf{C} \longrightarrow \mathbf{set}$$

preserves coproducts.

Theorem

Every family category is equivalent to the coproduct completion of its subcategory of connected objects.

We would like to study categories $\mathbf{C}$ which make $Fam(\mathbf{C})$ (finitely) complete and cocomplete.

Method for Producing the Fundamental Functor

We would like to construct the following functor (to match the original “fundamental functor"):

$$Fam(\mathbf{C}) \longrightarrow \mathbf{fset}$$

To do so, it is enough to consider:

$$\mathbf{C} \longrightarrow \mathbf{fset}$$

To construct the above, however, we need only consider:

$$Norm(\mathbf{C}) \longrightarrow \mathbf{fset}$$

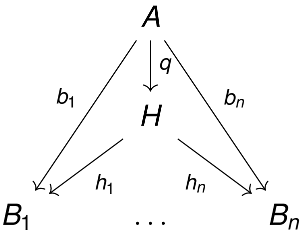
Where $Norm(\mathbf{C})$ is the subcategory of *normal objects*.

Normal Objects 1: Tables

Definition

A *table* is a wide pushout (which we will also call a *pre-table*) such that any map of wide pushouts leaving a table must be an isomorphism.

That is, given:



q must be an isomorphism.

Normal Objects 2

Definition

An object N in a category is *normal* if every table:

$$N \xleftarrow{\alpha} A \xrightarrow{\beta} N$$

necessarily has α and β isomorphisms.

Call a category **C** *normal* if every object is normal.

What is a Fundamental Functor?

Definition

For **C** normal, a functor $U : \mathbf{C} \longrightarrow \mathbf{fset}$ is *fundamental* if it is equipped with for each $N \in \mathbf{C}$, isomorphisms:

$$[\_]_N : U(N) \longrightarrow Hom_{\mathbf{C}}(N, N)$$
$$\{\_\}_N : Hom_{\mathbf{C}}(N, N) \longrightarrow U(N)$$

such that:

1. $\{[x]\} = x$
2. $[\{\alpha\}] = \alpha$
3. $U([x])(\{1_N\}) = x$
4. $[x]f = f[U(f)(\{1_N\})]^{-1}[U(f)(x)]$

Example of a Fundamental Functor

- Consider the *preorder collapse* of **C**:

$$\mathbf{C} \xrightarrow{\delta} \mathit{Preorder}(\mathbf{C})$$

Where *Preorder*(**C**) is the category whose objects are objects of **C** and given any two objects in **C**, if there is a map between them in **C**, then there is a map in *Preorder*(**C**).

Example of a Fundamental Functor Continued

- We say δ above is *split* if there is a functor

$$\lambda : \mathit{Preorder}(\mathbf{C}) \longrightarrow \mathbf{C}$$

such that:

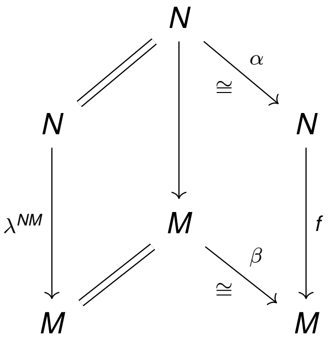
$$\begin{array}{ccc} \mathit{Preorder}(\mathbf{C}) & \xrightarrow{\lambda} & \mathbf{C} \\ & \searrow^{=} & \downarrow \delta \\ & & \mathit{Preorder}(\mathbf{C}) \end{array}$$

Example of a Fundamental Functor Continued

Given such a splitting, we can form the following functor:

$$\mathbf{C} \xrightarrow{U_\lambda} \mathbf{fset}$$

$$\begin{array}{ccc} N & \longmapsto & Table(N,N) \\ \Big\downarrow f & & \Big\downarrow \\ M & \longmapsto & Table(M,M) \end{array}$$



Significance of U_λ

U_λ is a fundamental functor!

Proposition

Given a fundamental functor $U : \mathbf{C} \longrightarrow \mathbf{fset}$, there exists a unique splitting λ such that $U \cong U_\lambda$.

Questions/Outlook

- What properties does **C** need so that the constructed fundamental functor above, U_λ, can be generalized to one on *Fam*(**C**) which agrees with Grothendieck's definition?
- What further results can we find about Galois categories using this method?
- How can this method be used in already existing examples? (eg. schemes)

THANK YOU!

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