Orbifolds as Manifolds

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Outline

Atlas Charts

- An effective (or, reduced) orbispace consists of a paracompact space M with an equivalence class of orbispace atlases.
- An orbispace chart $\mathcal{U} = \{\tilde{U}, G_U, \rho_U, \varphi_U\}$ consists of
  - $\tilde{U} \subseteq \mathbb{R}^n$, open (and contractible);
  - G_U a finite group;
  - a monomorphism $\rho_U \colon G_U \to \text{Homeo}(\tilde{U})$, defining a left action of G_U on $\tilde{U}$;
  - $\varphi_U \colon \tilde{U} \to \tilde{U}/G_U \cong U \subseteq M$, the quotient map into the orbitspace.

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Atlas Chart Embeddings

Given charts $\mathcal{U} = \{\tilde{U}, G_U, \rho_U, \varphi_U\}$ and $\mathcal{V} = \{\tilde{V}, G_V, \rho_V, \varphi_V\}$, an (atlas) chart embedding $\mathcal{U} \hookrightarrow \mathcal{V}$ consists of a pair

$$(\lambda\colon \tilde{U} \hookrightarrow \tilde{V}, \ell\colon G_U \hookrightarrow G_V)$$

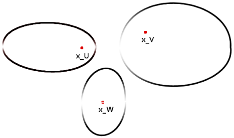
such that

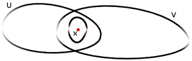
- $$\begin{array}{ccc} \tilde{U} & \xrightarrow{\lambda} & \tilde{V} \\ \downarrow{\scriptstyle \varphi_U} & & \downarrow{\scriptstyle \varphi_V} \\ U & \subseteq & V \end{array}$$
- $\lambda(g \cdot u) = \ell(g) \cdot \lambda(u)$ for $g \in G_U$ and $u \in \tilde{U}$.

Local Compatibility

For any two charts $\mathcal{U} = \{\tilde{U}, G_U, \rho_U, \varphi_U\}$ and $\mathcal{V} = \{\tilde{V}, G_V, \rho_V, \varphi_V\}$ with a point $x \in U \cap V \subseteq M$, there is a chart $\mathcal{W} = \{\tilde{W}, G_W, \rho_W, \varphi_W\}$ with $x \in W \subseteq U \cap V$ and atlas chart embeddings

$$\mathcal{U} \hookleftarrow \mathcal{W} \hookrightarrow \mathcal{V}$$



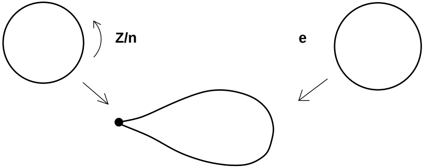


Equivalence of Atlases

Two atlases $\mathfrak{A}$ and $\mathfrak{B}$ for the space M are equivalent if they satisfy the following equivalent conditions:

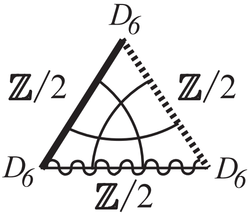
- There is a third atlas $\mathfrak{C}$ with $\mathfrak{C} \supseteq \mathfrak{A} \cup \mathfrak{B}$.
- There is a common refinement $\mathfrak{D}$.
- For each pair of charts $\mathcal{U} \in \mathfrak{A}$ and $\mathcal{V} \in \mathfrak{B}$ with a point $x \in U \cap V \subseteq M$ there exists an orbifold chart for a neighbourhood of x with chart embeddings into both $\mathcal{U}$ and $\mathcal{V}$.

Example 1: The Teardrop



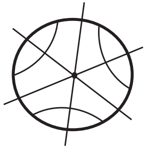
Example 2: The Triangular Billiard

The orbitspace:



Example 2: The Triangular Billiard

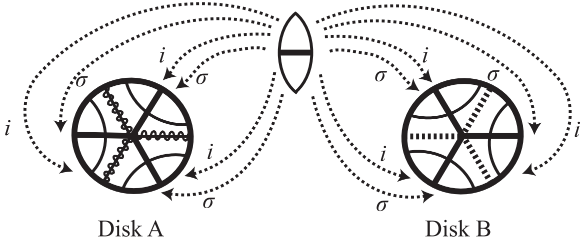
A chart for a corner:



with an action by the dihedral group D_6.

Example 2: The Triangular Billiard

Three charts with embeddings:



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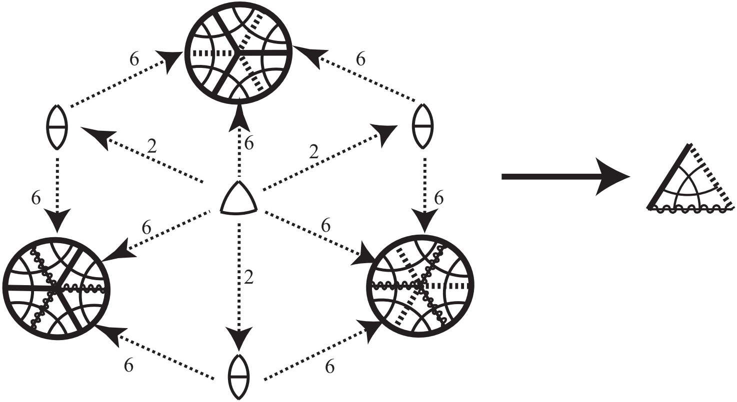

Embeddings and Homomorphisms

- Given a chart embedding $\lambda\colon \tilde{U} \hookrightarrow \tilde{V}$ (such that $\varphi_V \lambda = \varphi_U$) there is a unique (monic) group homomorphism $\ell\colon G_U \hookrightarrow G_V$ such that $\lambda(g \cdot u) = \ell(g) \cdot \lambda(u)$.
- So the atlas chart embeddings are determined by the λs.

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Embeddings as modules

- The set of embeddings $\left\{ \lambda\colon \tilde{U} \hookrightarrow \tilde{V};\ \begin{array}{ccc} \tilde{U} & \xrightarrow{\lambda} & \tilde{V} \\ {\scriptstyle \varphi_U}\downarrow & & \downarrow{\scriptstyle \varphi_V} \\ U & \subseteq & V \end{array} \right\}$ has a natural right action by G_U and a natural left action by G_V.
- The set of chart embeddings from a chart $\tilde{U}$ to itself is in 1-1 correspondence with G_U.

Effective Orbigroupoids

Given an orbifold atlas $\mathfrak{A}$, we can represent its data in a topological groupoid $\mathcal{G}(\mathfrak{A})$,

$$\mathcal{G}(\mathfrak{A})_1 \times_{\mathcal{G}(\mathfrak{A})_0} \mathcal{G}(\mathfrak{A})_1 \xrightarrow{m} \mathcal{G}(\mathfrak{A})_1 \xrightarrow{i} \mathcal{G}(\mathfrak{A})_1 \underset{t}{\overset{s}{\rightrightarrows}} \!\!\xleftarrow{u}\, \mathcal{G}(\mathfrak{A})_0,$$

as follows:

- The space of objects is $\mathcal{G}(\mathfrak{A})_0 = \coprod_{\mathcal{U}\in\mathfrak{A}} \tilde{U}$;
- The space of arrows has

$$s^{-1}(\tilde{U}) \cap t^{-1}(\tilde{U}) = G_U \times \tilde{U}$$

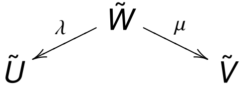
  with

$$s(g,u) = u \text{ and } t(g,u) = g \cdot u.$$

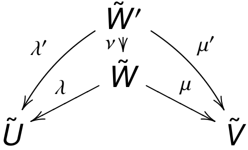
$\mathcal{G}(\mathfrak{A})_1$

$s^{-1}(\tilde{U}) \cap t^{-1}(\tilde{V}) = \lim_{\to} \tilde{W}$, the colimit of the diagram of spaces with

- **Objects:** charts $\mathcal{W}$ with chart embeddings

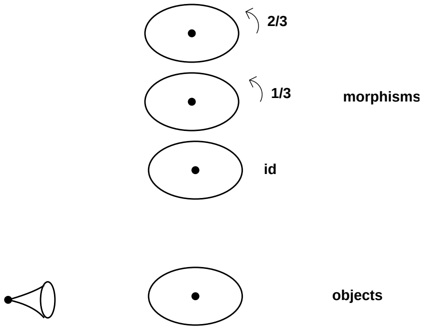


- **Arrows:** chart embeddings that commute with the embeddings of the objects:



The source map s is induced by the embeddings λ and the target map t is induced by the embeddings μ.

Example 0: The Cone Groupoid



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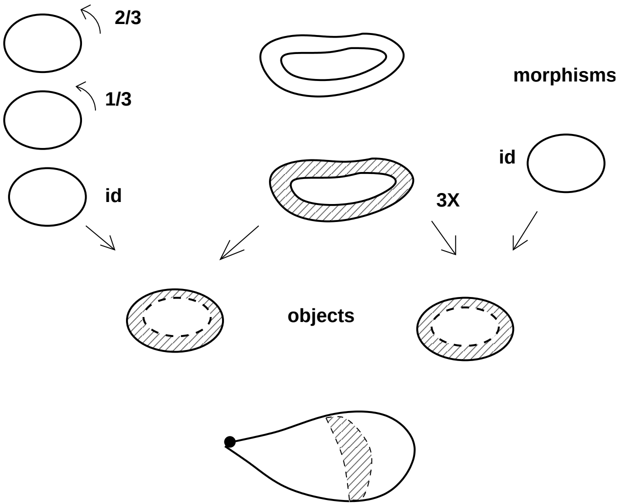
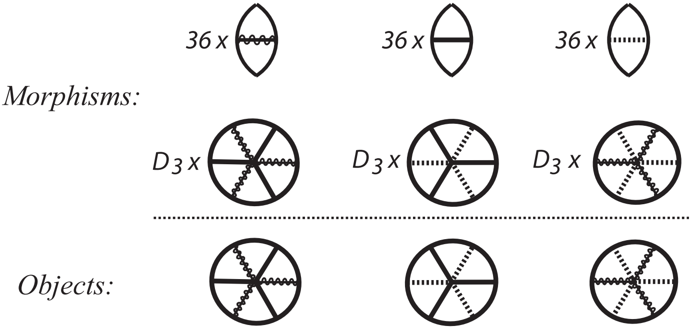


Example 2: The Triangular Billiard Groupoid



Notes

- The groupoid $\mathcal{G}(\mathfrak{A})$ is étale in the sense that all structure maps (s, t, u, i and m) are local homeomorphisms.
- The groupoid $\mathcal{G}(\mathfrak{A})$ is proper in the sense that $(s,t)\colon \mathcal{G}(\mathfrak{A})_1 \to \mathcal{G}(\mathfrak{A})_0 \times \mathcal{G}(\mathfrak{A})_0$ is proper (the preimage of any compact subset is compact).
- For any two charts $\tilde{U}$ and $\tilde{V}$, the arrows s and t in

$$\tilde{U} \xleftarrow{\ s\ } s^{-1}(\tilde{U}) \cap t^{-1}(\tilde{V}) \xrightarrow{\ t\ } \tilde{V}$$

are covering projections onto their images (with the groups G_V and G_U respectively acting as deck transformations).

Notes

- The groupoid $\mathcal{G}(\mathfrak{A})$ is effective.
- For any point $x \in \tilde{U} \subseteq \mathcal{G}(\mathfrak{A})_0$, $s^{-1}(x) \cap t^{-1}(x)$ is the isotropy group of x, i.e., the group $\{g \in G_U ; g \cdot x = x\}$.
- Equivalent atlases give rise to Morita equivalent groupoids,

$$\mathcal{G}(\mathfrak{A}) \longleftarrow \mathcal{K} \longrightarrow \mathcal{G}(\mathfrak{B}).$$

Definition

An orbigroupoid is a proper étale groupoid.

From Groupoids to Atlases: Translation Neighbourhoods

Given an effective orbigroupoid $\mathcal{G}$, we construct an effective orbispace atlas for its space of orbits $\mathcal{G}_0/\mathcal{G}_1$.

Step 1: The Charts

- $\mathcal{G}$ is étale and proper $\Rightarrow$ for each point $x \in \mathcal{G}_0$ there is a neighbourhood $\tilde{U}_x$ such that

$$s^{-1}(\tilde{U}_x) \cap t^{-1}(\tilde{U}_x) \cong \mathcal{G}_x \times \tilde{U}_x.$$

- We call $\tilde{U}_x$ a translation neighbourhood.
- Translation neighbourhoods form a basis for the topology on $\mathcal{G}_0$.

From Groupoids to Atlases: Translation Neighbourhood Embeddings

Step 2: Embeddings of Charts

- Given two translation neighbourhoods $\tilde{U}_x$ and $\tilde{U}_y$ with $U_x \subseteq U_y$, the maps in

$$\tilde{U}_x \xleftarrow{s} s^{-1}(\tilde{U}_x) \cap t^{-1}(\tilde{U}_y) \xrightarrow{t} \tilde{U}_y$$

  are covering projections, so we can obtain the chart embeddings from the connected components of the preimages.

Morita Equivalence and Atlas Equivalence

- Recall: Equivalent orbispace atlases give rise to Morita equivalent groupoids.
- Any two atlases obtained from the same orbigroupoid are equivalent as orbispace atlases.
- Any two atlases obtained from Morita equivalent orbigroupoids are equivalent as orbispace atlases.

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Atlases for Ineffective Orbispaces (Adem, Chen, Ruan, etc.)

An orbispace is *ineffective* (or, non-reduced) if the group actions are not effective.

- **Charts** for ineffective orbispaces are of the form $\mathcal{U} = \{\tilde{U}, G_U, \rho_U, \varphi_U\}$, where $\rho_U \colon G_U \to \mathrm{Homeo}(\tilde{U})$ is any group homomorphism.
- **Chart embeddings** $\mathcal{U} \hookrightarrow \mathcal{V}$ are pairs $(\lambda \colon \tilde{U} \hookrightarrow \tilde{V}, \ell \colon G_U \hookrightarrow G_V)$ as before with the additional property that

$$\ell|_{\mathrm{Ker}(\rho_U)} \colon \mathrm{Ker}(\rho_U) \xrightarrow{\sim} \mathrm{Ker}(\rho_V).$$

Ineffective Orbigroupoids

Ineffective orbigroupoids are just proper étale groupoids that are not required to be effective.

Atlases and Groupoids

We hope that:

- Orbispace charts correspond to translation neighbourhoods in a proper étale groupoid.
- Embeddings of charts $\mathcal{U} \hookrightarrow \mathcal{V}$ correspond to connected components of $s^{-1}(\tilde{U}) \cap t^{-1}(\tilde{V})$.

The Problem: Example 1

- Consider the orbispace described by one chart, D, the open unit disk in $\mathbb{R}^2$, with $G_D = \mathbb{Z}/2 \times \mathbb{Z}/2$, acting trivially.
- How many chart embeddings are there from this chart to itself?
- So what should its orbigroupoid presentation be?

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Our Goal

We want an atlas definition for orbispaces that agrees with the orbigroupoid description:

- an extension of the classical description for effective orbispaces;
- Morita equivalence classes of orbigroupoids should correspond to equivalence classes of atlases;
- with a notion of morphisms that will give rise to a (bi)equivalence of (bi)categories.

Classical Manifold Atlases

- With the classical notion of manifolds, you may have equivalent atlases, but there are no arrows from one to the other, there is just a refinement, or a common enlargement.
- Consequently, you cannot represent all smooth maps between two manifolds in terms of any given atlases.

The Manifold Construction

- The manifold construction defines a notion of atlas such that in the category of charts and atlases (the manifold completion of the category of charts) equivalent atlases are isomorphic.
- The trick behind this is that we allow ourselves to work with **partial maps**.
- In order to apply the manifold construction, we need a **join restriction category**.

Restriction categories

Definition

A **restriction category** $\mathbb{C}$ (of "charts") is a category with a **restriction structure** which assigns to each arrow $f\colon A \to B$, an arrow, $\overline{f}\colon A \to A$, such that:

R.1 $f\overline{f} = f$;

R.2 If $\mathrm{dom}(f) = \mathrm{dom}(g)$ then $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$;

R.3 If $\mathrm{dom}(f) = \mathrm{dom}(g)$ then $\overline{\overline{f}g} = \overline{f}\,\overline{g}$;

R.4 If $\mathrm{dom}(h) = \mathrm{cod}(f)$ then $\overline{h}f = f\overline{hf}$.

The Restriction Category for Ordinary Manifolds

- To obtain the classical notion of manifold, the restriction category $\mathbb{C}$ has open subsets of $\mathbb{R}^n$ as objects and partially defined smooth maps as morphisms.
- We can represent the morphisms in this category by spans

$$U \xleftarrow{\lambda} U' \xrightarrow{\varphi} V$$

where λ is a smooth embedding and φ is a smooth map.

Structure on the arrows

When two arrows agree wherever they are both defined, we say that they are compatible:

Definition

Two parallel maps $f, g \colon A \rightrightarrows B$ in a restriction category are **compatible**, written $f \smile g$, if $g\overline{f} = f\overline{g}$.

Restriction categories carry a natural enrichment over posets:

Definition

For two parallel maps $f, g \colon A \rightrightarrows B$ in a restriction category, we say that $f \leq g$ if $g\overline{f} = f$.

Restriction Idempotents and Partial Inverses

- An arrow $e\colon A \to A$ is a **restriction idempotent** if $\overline{e} = e$.
- The restriction idempotents on A form a semilattice.
- An arrow $f\colon A \to B$ is a **partial isomorphism** if there is an arrow $f^*\colon B \to A$ such that $ff^* = \overline{f^*}$ and $f^*f = \overline{f}$.
- An **inverse category** is a restriction category in which every arrow is a partial isomorphism.

Joins

- A set $S \subseteq \mathbb{C}(A, B)$ is called **compatible** if for any arrows $f, g \in S$, $f \smile g$.
- The restriction category $\mathbb{C}$ is a **join restriction category** if for each compatible family $S \subseteq \mathbb{C}(A, B)$, there is an arrow $\bigvee_{f \in S} f \in \mathbb{C}(A, B)$ such that:
  - $\bigvee_{f \in S} f$ is the join of S with respect to $\leq$ in $\mathbb{C}(A, B)$;
  - The join is stable with respect to composition: $(\bigvee_{f \in S} f)\, h = \bigvee_{f \in S}(fh)$.

It follows that:

- $k\, (\bigvee_{f \in S} f) = \bigvee_{f \in S}(kf)$;
- The restriction idempotents on A form a locale.

Joins and ordinary manifolds

The restriction category of charts for ordinary manifolds has joins of families of compatible maps:

$$\bigvee_{s\in S}\left(U \xleftarrow{\lambda_s} U'_s \xrightarrow{\varphi_s} V \right) = U \xleftarrow{\lambda} \bigcup_{s\in S} U'_s \xrightarrow{\varphi} V\,,$$

where λ is the obvious embedding and φ is the unique smooth map such that $\varphi|_{U'_s} = \varphi_s$.

Orbichart embeddings as modules

- Since the idea of using the group homomorphisms between the charts did not work, we want to use the idea that the embeddings between two charts should form a module with actions by the structure groups.
- This can be done in more than one way - here we will choose to stay fairly close to the structure we obtain directly from the orbigroupoids, and use topological modules/profunctors.

Action Groupoids

- Let a group G_X act on an open subset $X \subseteq \mathbb{R}^n$, then its action groupoid $G \ltimes X$ has space of objects X and space of arrows $G \times X$.
- The source map is given by projection and the target map by the action.
- Composition and inverses are induced by multiplication and inverses in the group G_X.

Topological Profunctors Between Action Groupoids

A topological profunctor $G_X \ltimes X \xrightarrow{U} G_Y \ltimes Y$ is given by a diagram of spaces $X \xleftarrow{p} U \xrightarrow{q} Y$ with

- A left action of G_Y, which makes q equivariant and preserves the fibers of p:

$$q(g \cdot u) = g \cdot q(u) \quad \text{and} \quad p(g \cdot u) = p(u)$$

- A right action of G_X, which makes p equivariant and preserves the fibers of q:

$$p(u \cdot g') = g'^{-1} \cdot p(u) \quad \text{and} \quad q(u \cdot g') = q(u)$$

- The two actions commute:

$$(g \cdot u) \cdot g' = g \cdot (u \cdot g')$$

Three Additional Conditions

For a topological profunctor $G_X \ltimes X \overset{U}{\nrightarrow} G_Y \ltimes Y$ to be an **orbispace profunctor**, we will further require that

- U is Hausdorff;
- the map $p\colon U \to X$ is open;
- the action of G_Y is **free and transitive** on the fibers of $p\colon U \to UX$ in the sense that it induces a homeomorphism of spaces,

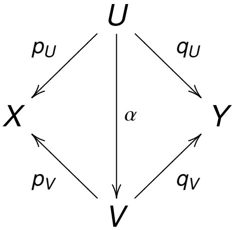
$$G_Y \times U \longrightarrow U \times_X U$$

$$(g, u) \longmapsto (g \cdot u, u)$$

So, whenever $p(u) = p(u')$ there is a unique $g \in G_Y$ such that $g \cdot u = u'$.

Maps between profunctors

Given two topological profunctors $U, V\colon G_X \ltimes X \nrightarrow G_Y \ltimes Y$, a map of profunctors $\alpha\colon U \to V$ is given by a continuous function α which is equivariant with respect to the actions of G_X and G_Y and commutes with the anchor maps,



Composition of Profunctors

Composition of topological profunctors

$$G_X \ltimes X \xrightarrow{U} G_Y \ltimes Y \xrightarrow{V} G_Z \ltimes Z$$

is given by a continuous version of the usual tensor construction:

- First take the pullback,

$$\begin{array}{ccc} V \times_Y U & \longrightarrow & U \\ \downarrow & & \downarrow q_U \\ V & \xrightarrow[p_V]{} & Y \end{array}$$

- The group G_Y acts on this space by $g' \cdot (v, u) = (v \cdot g'^{-1}, g' \cdot u)$.
- The composition profunctor $V \otimes_{G_Y} U$ is the orbitspace of $V \times_Y U$ under this action.

Units and Associativity

- The unit profunctor $G \ltimes U \longrightarrow\!\!\!\!\!\!\!\!+\;\; G \ltimes U$ is given by

$$U \xleftarrow{\pi_2} G \times U \xrightarrow{a} U$$

  where $a \colon G \times U \to U$ is given by $a(g, u) = g \cdot u$.
- The left and right actions of G on $G \times U$ are given by

$$g_1 \cdot (g, u) = (g_1 g, u) \quad \text{and} \quad (g, u) \cdot g_2 = (g g_2, g_2^{-1} \cdot u)$$

- Note that the composition of profunctors is only unitary and associative up to isomorphism.

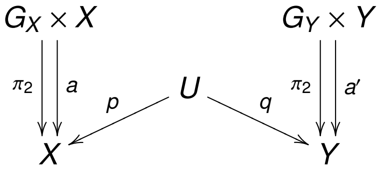
Orbispace Charts

The category **OrbiCharts** is defined as follows:

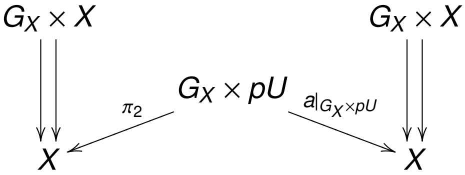
- **Objects** are actions groupoids $G_X \ltimes X$, where $X \subseteq \mathbb{R}^n$ is open (and contractible) and G is a finite group which acts on X.
- An **arrow** $G_X \ltimes X \overset{U}{\nrightarrow} G_Y \ltimes Y$ is a homeomorphism class of orbispace profunctors.

Restrictions for Orbispace Profunctors

The restriction $\overline{U}$ for an orbispace profunctor



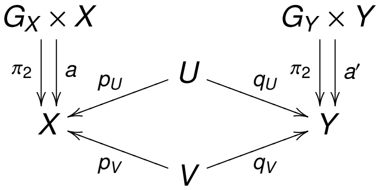
is given by



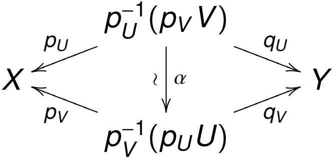
Note that pU is G_U-invariant, so this is well-defined.

Compatibility for Orbispace Profunctors

- Let



  be orbispace profunctors such that $V \otimes_{G_X} \overline{U} \cong U \otimes_{G_X} \overline{V}$.
- This means that there is a commutative diagram



  where α is equivariant with respect to both G_X and G_Y.

Binary Joins for Orbispace Profunctors

- When $\alpha\colon V \otimes_{G_X} \overline{U} \xrightarrow{\sim} U \otimes_{G_X} \overline{V}$, then

$$U \smile V = (U \amalg V)/(x \sim \alpha(x)),$$

  i.e, the following pushout,

$$\begin{array}{ccc} p^{-1}(qV) & \xrightarrow{\subseteq} & U \\ {\scriptstyle\alpha}\downarrow & & \downarrow \\ V & \longrightarrow & U \underset{p_U^{-1}(p_V V)}{\amalg} V = U \smile V \end{array}$$

- The relation $\{(x, \alpha(x)) | x \in U\}$ is closed in $(U \amalg V) \times (U \amalg V)$, so $(U \amalg V)/x \sim \alpha(x)$ is Hausdorff.
- $p \amalg q\colon U \smile V \to X$ is well-defined and open.
- The actions of G_X and G_Y on U and V induce well-defined actions on $U \smile V$ and make it an orbispace profunctor.

Arbitrary Joins for Orbispace Profunctors

To obtain the join $\bigvee_{i\in I}\left(G_X \ltimes X \xrightarrow{U_i} G_Y \ltimes Y\right)$ of profunctors represented by $X \xleftarrow{p_i} U_i \xrightarrow{q_i} Y$,

- take the colimit of the diagram with the spaces U_i for $i \in I$ and $p_i^{-1}(p_j U_j)$ for $i, j \in I$ and the arrows $U_i \hookleftarrow p_i^{-1}(p_j U_j) \xhookrightarrow{\alpha_{ij}} U_j$.
- The arrow into X is induced by the p_i.
- The arrow into Y is induced by the q_i.
- The actions of G_X and G_Y are induced by those on the U_i.

Partial Isomorphisms

Lemma

- *An orbispace profunctor* $G_X \ltimes X \overset{U}{\nrightarrow} G_Y \ltimes Y$*, with* $X \xleftarrow{p} U \xrightarrow{q} Y$ *is a partial isomorphism if and only if* G_X *acts freely and transitively on the fibers of* $U \xrightarrow{q} Y$*, i.e., it induces a homeomorphism* $G_X \times U \cong U \times_Y U$.
- *In this case, the partial inverse* $U^* \colon G_Y \ltimes Y \nrightarrow G_X \ltimes X$ *is given by* $Y \xleftarrow{q} U \xrightarrow{p} X$*, with the group actions defined by inverses:* $u \cdot' g = g^{-1} \cdot u$.

Atlases in Join Restriction Categories

Definition (Grandis)

Let $\mathbb{C}$ be a join restriction category. An **atlas** (C_i, φ_{ij}, I) in $\mathbb{C}$ consists of

- a family of objects C_i, $i \in I$ in $\mathbb{C}$;
- for each pair $i, j \in I$, a map $\varphi_{ji} \colon C_i \to C_j$;

such that for each triple $i, j, k \in I$,

Atl. 1 $\varphi_{ji}\varphi_{ii} = \varphi_{ji}$ (partial charts);

Atl. 2 $\varphi_{kj}\varphi_{ji} \leq \varphi_{ki}$ (cocycle condition);

Atl. 3 φ_{ij} is the partial inverse of φ_{ji}.

Atlases in **OrbiCharts**

Definition

An atlas in **OrbiCharts** consists of

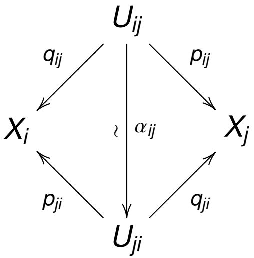
- a family of objects, $G_i \ltimes X_i$, for $i \in I$;
- for each pair $i, j \in I$, an orbispace profunctor $U_{ji} \colon G_i \ltimes X_i \nrightarrow G_j \ltimes X_j$;

such that for each triple $i, j, k \in I$,

- **OrbiAtl. 0** $\alpha_{ii} \colon U_{ii} \xrightarrow{\sim} \mathrm{Id}_{G_i \ltimes X_i}$, i.e., $\alpha_{ii} \colon U_{ii} \xrightarrow{\sim} G_i \times X_i$;
- (**OrbiAtl. 1** there are isomorphisms $\alpha_{jii} \colon U_{ji} \otimes_{G_i} U_{ii} \xrightarrow{\sim} U_{ji}$;)
- **OrbiAtl. 2** there are embeddings $\alpha_{kji} \colon U_{kj} \otimes_{G_j} U_{ji} \hookrightarrow U_{ki}$;
- **OrbiAtl. 3** U_{ij} is the partial inverse of U_{ji}, i.e., there are isomorphisms of topological profunctors, $\beta_{iji} \colon U_{ij} \otimes_{G_j} U_{ji} \xrightarrow{\sim} \overline{U}_{ji}$ and $\beta_{jij} \colon U_{ji} \otimes_{G_i} U_{ij} \xrightarrow{\sim} \overline{U}_{ij}$.

Inverses Revisited

For each triple $i, j \in I$,
OrbiAtl. 3 (improved version) There is an equivariant isomorphism



The atlas groupoid

Given an atlas $\mathfrak{X} = (G_i \ltimes X_i, [U_{ij}], I)$, with a choice of representatives $X_i \xleftarrow{p_{ij}} U_{ij} \xrightarrow{q_{ij}} X_j$, define the orbigroupoid $\mathcal{G}(\mathfrak{X})$ as follows:

- **Space of objects** $\mathcal{G}(\mathfrak{X})_0 = \amalg_{i \in I} X_i$
- **Space of arrows** $\mathcal{G}(\mathfrak{X})_1 = \amalg_{i,j \in I} U_{ij}$
- **Source and target maps** $s|_{U_{ij}} = p_{ij}$ and $t|_{U_{ij}} = q_{ij}$
- **The unit map** $u \colon \mathcal{G}(\mathfrak{X})_0 \to \mathcal{G}(\mathfrak{X})_1$ is defined on the component X_i by $u(x) = \alpha_{ii}^{-1}(x, e_{G_i})$.
- **Composition** for $(f', f) \in U_{kj} \times_{X_j} U_{ji}$, define $m(f', f) = \alpha_{kji}(f' \otimes f) \in U_{ik}$.
- **Inverses** for $f \in U_{ij}$, $i(f) \in U_{ji}$ is defined by $i(f) = \alpha_{ij}(f)$.

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Is this a groupoid?

Additional conditions to obtain strict units, associativity:

- Units: $\alpha_{ji} \circ (\alpha_{ii}^{-1}(e_{G_i}, q_{ij}(-), -) = \mathrm{id}_{U_{ij}}$
- Associativity:

$$\begin{array}{ccccc}
(U_{lk} \otimes_{G_k} U_{kj}) \times_{X_j} U_{ji} & \xrightarrow{\alpha_{lkj}\times U_{ji}} & U_{lj} \times_{X_j} U_{ji} & \twoheadrightarrow & U_{lj}\otimes_{G_j} U_{ji} \\
\uparrow & & & & \downarrow \alpha_{lji} \\
U_{lk}\times_{X_k} U_{kj}\times_{X_j} U_{ji} & & & & U_{li} \\
\downarrow & & & & \uparrow \alpha_{lki} \\
U_{lk}\times_{X_k} U_{kj}\otimes_{G_j} U_{ji}) & \xrightarrow[U_{lk}\times\alpha_{kji}]{} & U_{lk}\times_{X_k} U_{ki} & \twoheadrightarrow & U_{lk}\otimes_{G_k} U_{ki}
\end{array}$$

From Orbigroupoids to Atlases

Given an orbigroupoid $\mathcal{G}$,

- take a collection of translation neighbourhoods $X_i \subseteq \mathcal{G}_0$, with structure groups G_i, which essentially covers $\mathcal{G}_0$ (it meets every orbit);
- then $s^{-1}(X_i) \cap t^{-1}(X_j)$ has a left action of G_j and a right action of G_i;
- furthermore, $X_i \xleftarrow{s} s^{-1}(X_i) \cap t^{-1}(X_j) \xrightarrow{t} X_j$ carries the structure of an orbispace profunctor which is a partial isomorphism.
- Composition in $\mathcal{G}$ gives rise to the required isomorphisms to make this an orbichart atlas.

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Atlas Maps in Join Restriction Categories

Definition

Let (C_i, φ_{ij}, I) and (D_k, ψ_{kl}, K) be atlases in a join restriction category $\mathbb{C}$. An **atlas map**

$$A\colon (C_i, \varphi_{ij}, I) \to (D_k, \psi_{kl}, K)$$

consists of a family of maps $A_{ki}\colon C_i \to D_k$ ($i \in I$, $k \in K$), such that

- AtlM. 1 $A_{ki}\varphi_{ii} = A_{ki}$;
- AtlM. 2 $A_{kj}\varphi_{ji} \leq A_{ki}$;
- AtlM. 3 $\psi_{hk}A_{ki} = A_{hi}\overline{A}_{ki}$.

The atlas map is a total map when $\bigvee_{k\in K} \overline{A}_{ki} = \overline{\varphi}_{ii}$.

Maps Between OrbiChart Atlases

Definition

Let $(G_i \ltimes X_i, U_{ij}, I)$ and $(H_k \ltimes Y_k, V_{kl}, K)$ be atlases in the join restriction category **OrbiCharts.** An **atlas map**

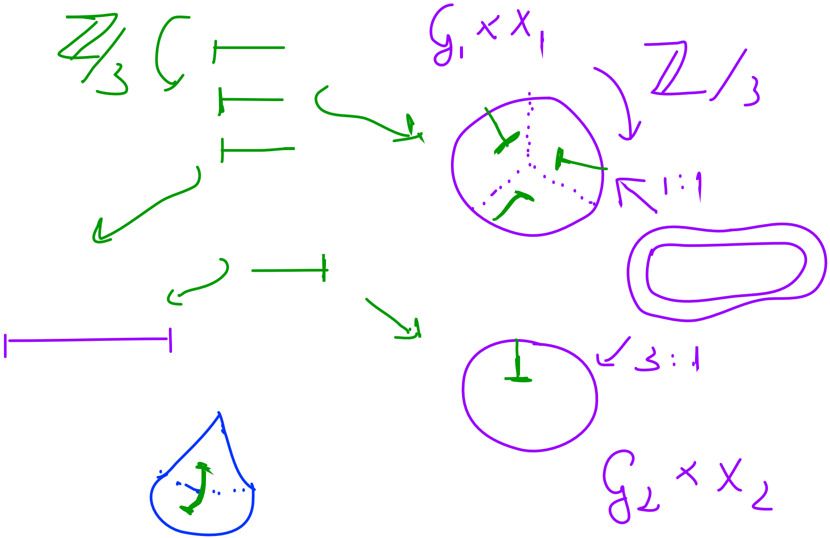
$$A\colon (G_i \ltimes X_i, U_{ij}, I) \to (H_k \ltimes Y_k, V_{kl}, K)$$

consists of a family of orbispace profunctors $A_{ki}\colon G_i \ltimes X_i \to H_k \ltimes Y_k$ $(i \in I, k \in K)$, such that

- OrbiAtlM. 1 $A_{ki} \otimes_{G_i} U_{ii} \cong A_{ki}$;
- OrbiAtlM. 2 $A_{kj} \otimes_{G_j} U_{ji} \leq A_{ki}$;
- OrbiAtlM. 3 $V_{hk} \otimes_{H_k} A_{ki} \cong A_{hi} \otimes_{G_i} \overline{A}_{ki}$.

The atlas map is a total map when $\bigvee_{k\in K} \overline{A}_{ki} = \overline{U}_{ii}$.

Example: Paths in the Teardrop



Hilsum Skandalis Maps

Definition

Let $\mathcal{G}$ and $\mathcal{H}$ be two orbigroupoids. A **Hilsum-Skandalis map**

$$M\colon \mathcal{G} \nrightarrow \mathcal{H}$$

is a topological profunctor

$$\mathcal{G}_0 \xleftarrow{p} M \xrightarrow{q} \mathcal{H}_0$$

(i.e., M has a left action of $\mathcal{H}$ which keeps the fibers of p invariant and it has a right action of $\mathcal{G}$ which keeps the fibers of q invariant), such that p is an open surjection and the action of $\mathcal{H}$ is free and transitive on the fibers.

Atlas Maps and Hilsum Skandalis Maps

- Hilsum Skandalis maps correspond to atlas maps which are total.
- M is a Morita equivalence when q is an open surjection and the action of $\mathcal{G}$ is free and transitive on the fibers of p.
- Isomorphic atlases give rise to Morita equivalent atlas groupoids.
- Atlases obtained from Morita equivalent groupoids are isomorphic.

Further Research

- We really want to view orbispaces as a ‘manifolds’ in a restriction bicategory.
- The manifolds obtained from the manifold construction are not necessarily Hausdorff or of a well-defined dimension. One can obtain the traditional manifolds by idempotent splitting. Can this be generalized to the category **OrbiCharts** to obtain the usual orbispaces?