On Double Inverse Semigroups

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In the beginning. . .

In his 2006 paper "Note on commutativity in double semigroups and two-fold monoidal categories", Kock introduced the notion of a double semigroup, along with some commutativity properties of them. In particular, he defines double inverse semigroups.

Definition

A *double semigroup* $(S, \circledcirc, \odot)$ is a set equipped with two associative binary operations satisfying the *middle-four interchange law*: for all $a, b, c, d \in S$,

$$(a \circledcirc b) \odot (c \circledcirc d) = (a \odot c) \circledcirc (b \odot d).$$

- Horizontal product: $a \circledcirc b = \begin{array}{|c|c|}\hline a & b \\ \hline\end{array}$.
- Vertical product: $a \odot b = \begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array}$.
- Middle-four:

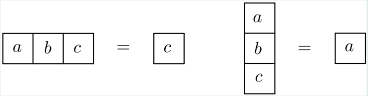
$$\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}$$

Example

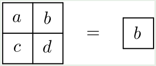
Any set D can be made into a double semigroup by equipping it with left and right projection:

$$a \odot b = a \text{ and } a \circledcirc b = b.$$

Associative:



Middle-four interchange law:



Definition

Given an element, x in a semigroup $(S, \odot)$, x said to have an *inverse* $x^{\odot}$ if

$$x = x \odot x^{\odot} \odot x \text{ and } x^{\odot} = x^{\odot} \odot x \odot x^{\odot}.$$

A semigroup is said to be an *inverse semigroup* if every element has a unique inverse. A double semigroup is said to be inverse if both of its operations are.

Note

$x \odot x^{\odot}$ *and* $x^{\odot} \odot x$ *are idempotents:*

- $(x \odot x^{\odot}) \odot (x \odot x^{\odot}) = (x \odot x^{\odot} \odot x) \odot x^{\odot} = x \odot x^{\odot}$
- $(x^{\odot} \odot x)(x^{\odot} \odot x) = (x^{\odot} \odot x \odot x^{\odot}) \odot x = x^{\odot} \odot x$

Theorem (Kock)

Double inverse semigroups are commutative.

In the comments of his LaTeX source code, Kock mentions that he does not have any "significant" examples of a double inverse semigroup. We aim to either find one, or to characterise double inverse semigroups.

Coming soon:

- Explore Lawson's correspondence between inductive groupoids and inverse semigroups given by a pair of constructions.
- Define double inductive groupoids.
- Extend these constructions to double inductive groupoids and double inverse semigroups and establish an analogous correspondence.

A quick notational note:

If $f : A \to B$ is an arrow in a category:

Notation

- *Domain of f* : $f\mathrm{dom} = A$.
- *Codomain of f* : $f\mathrm{cod} = B$.
- *Denote the composite*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

as $f;g$ *or* fg.

Definition

Let $(G, \bullet)$ be a groupoid and let $\leq$ be a partial order defined on the arrows of G. We call $(G, \bullet, \leq)$ and *ordered groupoid* whenever the following conditions are satisfied:

– If $x \leq y$, then $x^{-1} \leq y^{-1}$.

– If $x \leq y$, $u \leq v$, then $xu \leq yv$.

Note

Identification of identity arrows with objects:

- *Gives $\leq$ on objects*

Definition (cont'd)

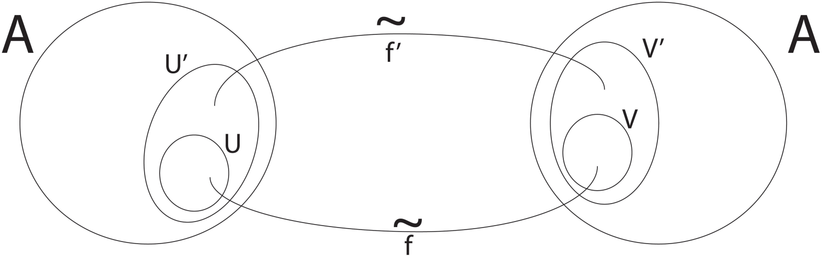
- Let $f \in G_1$ and let e be an object in G such that $e \leq f\mathrm{dom}$. Then there is a unique element $(e_*|f) \in G_1$, called the restriction of f by e, such that $(e_*|f) \leq f$ and $(e_*|f)\mathrm{dom} = e$.
- Let $f \in G_1$ and let e be an object in G such that $e \leq f\mathrm{cod}$. Then there is a unique element $(f|_*e) \in G_1$, called the corestriction of f by e, such that $(f|_*e) \leq f$ and $(f|_*e)\mathrm{cod} = e$.

$$\begin{array}{ccc} f\mathrm{dom} & \xrightarrow{\;f\;} & f\mathrm{cod} \\ \rotatebox{90}{$\leq$} & & \\ e & \xrightarrow[(e_*|f)]{} & (e_*|f)\mathrm{cod} \end{array}$$

Example

Let A be a set. Construct an inductive groupoid with the following data:

- Objects : $\mathcal{P}A$
- Arrows: Partial isomorphisms $f : U \xrightarrow{\sim} V$ between subsets $U, V \in \mathcal{P}A$
- $(f : U \to V) \leq (f' : U' \to V')$ if and only if $U \subseteq U'$ and f' restricted to U (as functions) is f.


A
U′
U
~
f′
~
f
V′
V
A

Definition

An ordered groupoid G is an *inductive groupoid* if its objects form a meet-semilattice.

Inductive Groupoids from Inverse Semigroups

Construction

Given an inverse semigroup $(S, \odot)$ *with the natural partial ordering* $\leq$, *define an inductive groupoid,* $(IG(S), \bullet)$, *with the following data:*

Objects: *idempotents of* S; $IG(S)_0 = E(S)$.

Arrows: *elements of* S.

Construction (cont'd)

Arrows: *elements of* S.

- $\mathrm{sdom} = s \odot s^{\odot}$
- $\mathrm{scod} = s^{\odot} \odot s$
- *If* $a^{\odot} \odot a = b \odot b^{\odot}$, *define* $a \bullet b = a \odot b$
- *Every arrow is an isomorphism with* $a^{-1} = a^{\odot}$
- $(a|_{*}e) = a \odot e$
- $(e_{*}|a) = e \odot a$

Inverse Semigroups from Inductive Groupoids

Construction

Given an inductive groupoid $(G, \bullet, \leq, \wedge)$*, construct an inverse semigroup* $(IS(G), \odot)$ *with* $IS(G) = G_1$ *and, for any* $a, b \in S$*,*

$$a \odot b = (a|_* a\mathrm{cod} \wedge b\mathrm{dom}) \bullet (a\mathrm{cod} \wedge b\mathrm{dom}_*|b).$$

An Isomorphism of Categories

Notation

Denote the category of inverse semigroups and semigroup homomorphisms as **IS**. *Denote the category of inductive groupoids and inductive functors as* **IG**.

Theorem (Lawson)

The categories **IG** *and* **IS** *are isomorphic.*

GOAL: Double this theorem

Definition

A *double category* $\mathcal{D}$ consists of the following data:

- A collection $\mathcal{D}_0$ of objects.
- A collection $\mathrm{Ver}(\mathcal{D})$ of vertical arrows.
  Associative and unitary composition:

$$A \xrightarrow{f} B \xrightarrow{g} C \quad = \quad A \xrightarrow{f \bullet g} C$$

$$A \xrightarrow{1_A} A \xrightarrow{f} B \quad = \quad A \xrightarrow{f} B \quad = \quad A \xrightarrow{f} B \xrightarrow{1_B} B$$

Definition (cont'd)

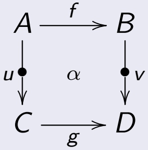
- A collection $\mathrm{Hor}(\mathcal{D})$ of horizontal arrows.
  Associative and unitary composition:

$$A \xrightarrow{f} B \xrightarrow{g} C \quad = \quad A \xrightarrow{f \circ g} C$$

$$A \xrightarrow{\mathrm{id}_A} A \xrightarrow{f} B \quad = \quad A \xrightarrow{f} B \quad = \quad A \xrightarrow{f} B \xrightarrow{\mathrm{id}_B} B$$

Definition (Cont'd)

– A collection $\mathrm{Dbl}(\mathcal{D})$ of double cells. A double cell α has the following form:



- A, B, C and D are objects of $\mathcal{D}$.
- Horizontal domain and codomain:

$$\alpha\mathrm{hdom} = u \text{ and } \alpha\mathrm{hcod} = v$$

- Vertical domain and codomain:

$$\alpha\mathrm{vdom} = f \text{ and } \alpha\mathrm{vcod} = g$$

Definition (cont'd)

These double cells must come together with:

- An associative and unitary horizontal composition, $\circ$.
- An associative and unitary vertical composition, $\bullet$.
- Horizontal and vertical composition of double cells must satisfy the middle-four interchange law. That is, for any $\alpha, \beta, \gamma, \delta \in \mathrm{Dbl}(\mathcal{D})$,

$$(\alpha \bullet \beta) \circ (\gamma \bullet \delta) = (\alpha \circ \gamma) \bullet (\beta \circ \delta).$$

Definition

A *double inductive groupoid*, denoted DIG,

$$\mathcal{G} = (\mathrm{Obj}(\mathcal{G}), \mathrm{Ver}(\mathcal{G}), \mathrm{Hor}(\mathcal{G}), \mathrm{Dbl}(\mathcal{G}))$$

is a double groupoid (i.e., a double category in which every vertical arrow, horizontal arrow and double cell is an isomorphism) such that :

Definition (cont'd)

$(\mathrm{Ver}(\mathcal{G}), \mathrm{Dbl}(\mathcal{G}))$ is an inductive groupoid.

- Composition: horizontal composition, $\circ$.
- Partial order : $\leq$.
- Meet of vertical arrows e and f : $e \wedge_h f$.
- For a double cell α and a vertical arrow e with $e \leq \alpha\mathrm{hdom}$, horizontal restriction : $(e_*|\alpha)$.
- If $e \leq \alpha\mathrm{hcod}$, horizontal corestriction: $(\alpha|_* e)$.

Definition (cont'd)

$(\mathrm{Hor}(\mathcal{G}), \mathrm{Dbl}(\mathcal{G}))$ is an inductive groupoid.

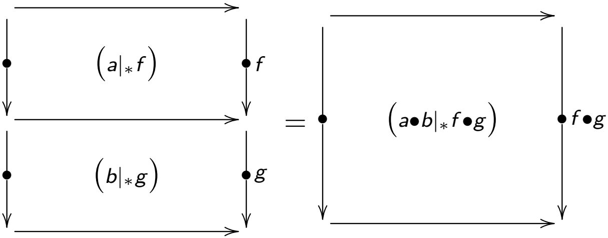
- Composition: vertical composition, $\bullet$.
- Partial order : $\lesssim$.
- Meet of horizontal arrows e and f : $e \wedge_v f$.
- For a double cell α and a horizontal arrow e with $e \lesssim \alpha \mathrm{vdom}$, vertical restriction : $[e_* | \alpha]$.
- If $e \lesssim \alpha \mathrm{vcod}$, vertical corestriction: $[\alpha |_* e]$.

Definition (cont'd)

If a, b are double cells, f', g' are horizontal arrows and f, g are vertical arrows, the following laws about restrictions and corestrictions preserving composition hold:

1. $\left(a \bullet b|_* f \bullet g\right) = \left(a|_* f\right) \bullet \left(b|_* g\right)$.
2. $\left[a \circ b|_* f' \circ g'\right] = \left[a|_* f'\right] \circ \left[b|_* g'\right]$.
3. $\left(f \bullet g_*|a \bullet b\right) = \left(f_*|a\right) \bullet \left(g_*|b\right)$.
4. $\left[f' \circ g'_*|a \circ b\right] = \left[f'_*|a\right] \circ \left[g_*|b\right]$.

$\left(a \bullet b|_* f \bullet g\right) = \left(a|_* f\right) \bullet \left(b|_* g\right)$



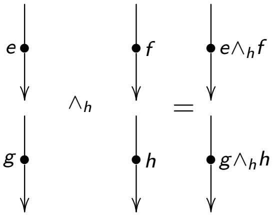
Definition (cont'd)

If e, f, g and h are horizontal arrows and e', f', g' and h' are vertical arrows, the following laws about composition and meets satisfying middle-four hold:

(a) $(e \wedge_v f) \circ (g \wedge_v h) = (e \circ g) \wedge_v (f \circ h)$.

(b) $(e' \wedge_h f') \bullet (g' \wedge_h h') = (e' \bullet g') \wedge_h (f' \bullet h')$.

$(e \wedge_h f) \bullet (g \wedge_h h) = (e \bullet g) \wedge_h (f \bullet h)$

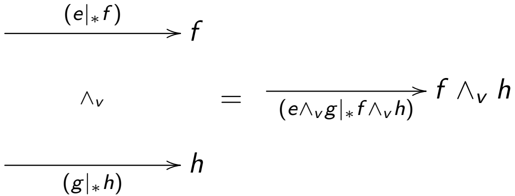


Definition (cont'd)

If e and g are horizontal arrows f and h are objects, then the following rule about corestrictions and meets satisfying middle-four holds:

$$(e|_* f) \wedge_v (g|_* h) = (e \wedge_v g|_* f \wedge_v h)$$

$(e|_* f) \wedge_v (g|_* h) = (e \wedge_v g|_* f \wedge_v h)$



Similarly,

(a) $(e|_* f) \wedge_v (g|_* h) = (e \wedge_v g|_* f \wedge_v h)$.

(b) $[e'|_* f'] \wedge_h [g'|_* h'] = [e' \wedge_h g'|_* f' \wedge_h h']$.

(c) $(e_*|f) \wedge_v (g_*|h) = (e \wedge_v g_*|f \wedge_v h)$.

(d) $[e'_*|f'] \wedge_h [g'_*|h'] = [e' \wedge_h g'_*|f' \wedge_h h']$.

Definition (cont'd)

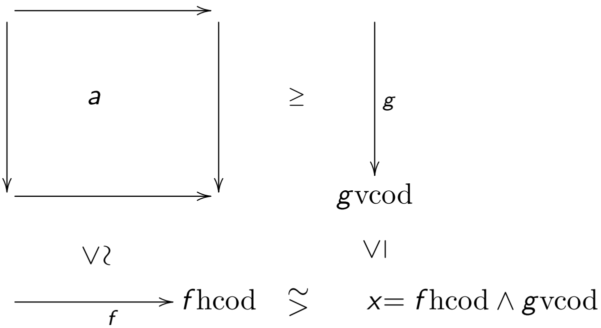
If a is a double cell, f a horizontal arrow, g a vertical arrow and x an object such that

$$f \lesssim a\mathrm{vcod}$$
$$g \leq a\mathrm{hcod}$$
$$x = f\mathrm{hcod} \wedge g\mathrm{vcod},$$

then the following middle-four law about vertical and horizontal corestrictions holds:

$$([a|_* f]|_* [g|_* x]) = [(a|_* g)|_* (f|_* x)]$$

$([a|_* f]|_*[g|_* x]) = [(a|_* g)|_*(f|_* x)]$



Similarly,

(a) $[(a|_* g)|_*(f|_* x)] = ([a|_* f]|_*[g|_* x])$.
(b) $([x_*|g]_*|[f_*|a]) = [(x_*|f)_*|(g_*|a)]$.
(c) $[(x_*|f)_*|(g_*|a)] = ([x_*|g]|[f_*|a])$.

Definition (cont'd)

If e, f, g and h are objects, the following law about meets satisfying middle-four holds:

$$(e \wedge_h f) \wedge_v (g \wedge_h h) = (e \wedge_v g) \wedge_h (f \wedge_v h).$$

Definition (cont'd)

If a is a double cell, e a vertical arrow and e' a horizontal arrow, then the following laws about domains and codomains preserving restrictions and corestrictions hold:

(a) $(a|_* e)\mathrm{vdom} = (a\mathrm{vdom}|_* e\mathrm{vdom})$.

(b) $(a|_* e)\mathrm{vcod} = (a\mathrm{vcod}|_* e\mathrm{vcod})$.

(c) $(e_*|a)\mathrm{vdom} = (e\mathrm{vdom}_*|a\mathrm{vdom})$.

(d) $(e_*|a)\mathrm{vcod} = (e\mathrm{vcod}_*|a\mathrm{vcod})$.

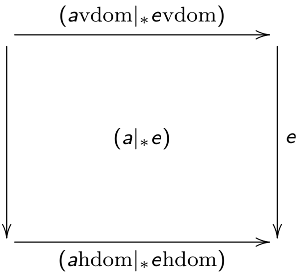
(e) $[a|_* e']\mathrm{hdom} = [a\mathrm{hdom}|_* e'\mathrm{hdom}]$.

(f) $[a|_* e']\mathrm{hcod} = [a\mathrm{hcod}|_* e'\mathrm{hcod}]$.

(g) $[e'_*|a]\mathrm{hdom} = [e'\mathrm{vdom}_*|a\mathrm{hdom}]$.

(h) $[e'_*|a]\mathrm{hcod} = [e'\mathrm{hcod}_*|a\mathrm{hcod}]$.

$(a|_* e)\mathrm{vdom} = (a\mathrm{vdom}|_* e\mathrm{vdom})$
$(a|_* e)\mathrm{hdom} = (a\mathrm{hdom}|_* e\mathrm{hdom})$



Double Inductive Groupoids from Double Inverse Semigroups

Construction (DIG)

Given a double inverse semigroup $(S, \circledcirc, \odot)$, *we construct a double inductive groupoid*

$$\mathbf{DIG}(S) = (\mathbf{DIG}(S)_0, \mathrm{Ver}(\mathbf{DIG}(S)), \mathrm{Hor}(\mathbf{DIG}(S)), \mathrm{Dbl}(\mathbf{DIG}(S)))$$

as follows:
Objects: $\mathbf{DIG}(S)_0 = E(S, \circledcirc) \cap E(S, \odot)$.

Construction (DIG cont'd)

Vertical arrows: $\mathrm{Ver}(\mathbf{DIG}(S)) = E(S, \circledcirc)$. *Let u and v be any two vertically composable arrows:*

- $u\mathrm{vdom} = u \odot u^{\odot}$
- $u\mathrm{vcod} = u^{\odot} \odot u$
- *Vertical composition:* $u \bullet v = u \odot v$

Horizontal arrows: $\mathrm{Hor}(\mathbf{DIG}(S)) = E(S, \odot)$. *Let f and g be any two horizontally composable arrows:*

- $f\mathrm{hdom} = f \circledcirc f^{\circledcirc}$
- $f\mathrm{hcod} = f^{\circledcirc} \circledcirc f$
- *Horizontal composition:* $f \circ g = f \circledcirc g$

Construction (DIG cont'd)

$\mathrm{Dbl}(\mathbf{DIG}(S)) = S(\circledcirc, \odot)$. *Let* a, b *be any two horizontally composable double cells.*

Horizontally:

- $\mathrm{ahdom} = a \circledcirc a^{\circledcirc}$
- $\mathrm{ahcod} = a^{\circledcirc} \circledcirc a$
- *Horizontal composition:* $a \circ b = a \circledcirc b$
- *Horizontal partial order:* $a \leq b$ *iff* $a = \mathrm{id}_e \circledcirc b$ *for some vertical arrow* e
- *Horizontal meet of two vertical arrows* e *and* f : $e \wedge_h f = e \circledcirc f$
- *If we have a vertical arrow* $e \leq \mathrm{ahcod}$, *define* $(a|_* e) = a \circledcirc e$
- *If* $e \leq \mathrm{ahdom}$, *define* $(e_*|a) = e \circledcirc a$.

Construction (DIG cont'd)

$\mathrm{Dbl}(\mathbf{DIG}(S)) = S(\circledcirc, \odot)$. *Let* a, b *be any two vertically composable double cells.*

Vertically:

- $\mathrm{avdom} = a \odot a^{\odot}$
- $\mathrm{avcod} = a^{\odot} \odot a$
- *Vertical composition:* $a \bullet b = a \odot b$
- *Vertical partial order:* $a \lesssim b$ *iff* $a = 1_e \odot b$ *for some horizontal arrow* e
- *Vertical meet of two horizontal arrows* e *and* f : $e \wedge_v f = e \odot f$
- *If we have a horizontal arrow* $e \lesssim \mathrm{avcod}$, *define* $[a|_* e] = a \odot e$
- *If* $e \lesssim \mathrm{avdom}$, *define* $[e_*|a] = e \odot a$

Double Inductive Groupoids from Double Inverse Semigroups

Theorem

If $S(\circledcirc, \odot)$ is a double inverse semigroup, then $\mathbf{DIG}(S)$, *as constructed in Construction DIG, is a double inductive groupoid.*

Double Inverse Semigroups from Double Inductive Groupoids

Construction (DIS)

Given a double inductive groupoid

$$\mathcal{G} = (\mathrm{Obj}(\mathcal{G}), \mathrm{Ver}(\mathcal{G}), \mathrm{Hor}(\mathcal{G}), \mathrm{Dbl}(\mathcal{G})),$$

we construct a double inverse semigroup $\mathbf{DIS}(\mathcal{G}) = (S, \circledcirc, \odot)$ *as follows:*

- *Its elements are the double cells of* $\mathcal{G}$; $S = \mathrm{Dbl}(\mathcal{G})$.

Construction (DIS cont'd)

- *For any* $a, b \in S$, *define*

$$a \circledcirc b = (a|_* a\mathrm{hcod} \wedge_h b\mathrm{hdom}) \circ (a\mathrm{hcod} \wedge_h b\mathrm{hdom}_*|b)$$

- *For any* $a, b \in S$, *define*

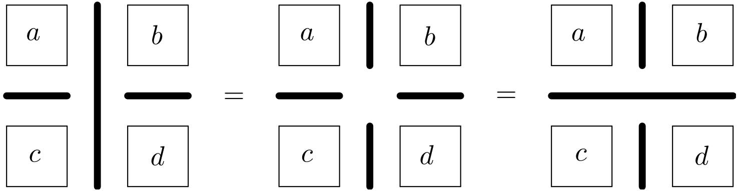
$$a \odot b = [a|_* a\mathrm{vcod} \wedge_v b\mathrm{vdom}] \bullet [a\mathrm{vcod} \wedge_v b\mathrm{vdom}_*|b]$$

Double Inverse Semigroups from Double Inductive Groupoids

Theorem

If $\mathcal{G}$ is a double inductive groupoid, then $\mathbf{DIS}(\mathcal{G})$, *as constructed in Construction DIS, is a double inverse semigroup.*

Most of the work in proving this is in checking that the middle-four interchange law is satisfied.



An Isomorphism of Categories

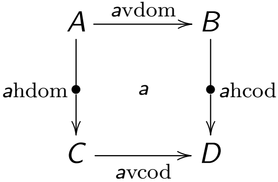
Notation

We denote the category of double inductive groupoids with double inductive functors as **DIG** *and we denote the category of double inverse semigroups with double semigroup homomorphisms as* **DIS**.

Theorem

There exists an isomorphism of categories between **DIG** *and* **DIS**.

Consider a double cell in a double inductive groupoid



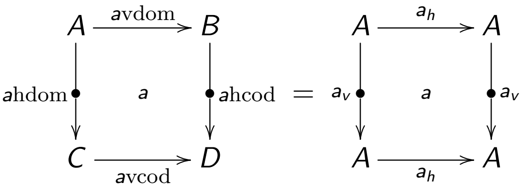
Recall that domains and codomains may be written as semigroup products and that double inverse semigroups are commutative. Then

- $a_h := a\mathrm{hdom} = a \circledcirc a^{\circledcirc} = a^{\circledcirc} \circledcirc a = a\mathrm{hcod}$
- $a_v := a\mathrm{vdom} = a \odot a^{\odot} = a^{\odot} \odot a = a\mathrm{hcod}$

Similarly, the domain and codomain of a vertical or horizontal arrows are equal, so that

- $A = a_h\mathrm{hdom} = a_h\mathrm{hcod}$
- $A = a_v\mathrm{vdom} = a_v\mathrm{vcod}$

Ultimately, a is of the form



Let $\mathcal{G}$ be a double inductive groupoid and let A be an object of $\mathcal{G}$. Then there is a natural collection of double cells

$$(A)\mathcal{S}_{\mathcal{G}} = \left\{ a \in \mathrm{Dbl}(\mathcal{G}) \;\middle|\; \begin{array}{ccc} A & \xrightarrow{a_h} & A \\ {\scriptstyle a_v}\big\downarrow\bullet & a & \bullet\big\downarrow{\scriptstyle a_v} \\ A & \xrightarrow[a_h]{} & A \end{array} \right\}$$

- Recall: Objects of double inductive groupoids are idempotent with respect to both operations of its corresponding double inverse semigroup.
- Double inverse semigroups are commutative.
- $$(a \circledcirc b) \odot (a \circledcirc b) = (a \odot a) \circledcirc (b \odot b) = a \circledcirc b$$
- $$\begin{aligned} a \odot b &= (a \odot b) \circledcirc (a \odot b) \\ &= (a \odot b) \circledcirc (b \odot a) \\ &= (a \circledcirc b) \odot (b \circledcirc a) \\ &= (a \circledcirc b) \odot (a \circledcirc b) \\ &= a \circledcirc b. \end{aligned}$$

Lemma

The vertical and horizontal order relations on the objects of a double inductive groupoid coincide.

Theorem

These one-object double inductive groupoids are precisely Abelian groups.

Proposition

Let $\mathcal{G}$ be a double inductive groupoid. If A and B are objects in $\mathcal{G}$ with $A \leq B$, then there is an Abelian group homomorphism

$$\varphi_{A \leq B} : (B)\mathcal{S}_{\mathcal{G}} \to (A)\mathcal{S}_{\mathcal{G}}.$$

This discussion results in an **Ab**−valued presheaf

$$\mathcal{S}_{\mathcal{G}} : \mathrm{Obj}(\mathcal{G})^{\mathrm{op}} \to \textbf{Ab}.$$

Theorem

Arbitrary double inverse semigroups are **Ab**−*valued presheaves over meet-semilattices.*

Construction

If $P : L^{\mathrm{op}} \to \mathbf{Ab}$ is a presheaf of Abelian groups on a meet-semilattice, define a double inductive groupoid $\mathcal{G} = PF'$ with the following data:

Objects: $\mathrm{Obj}(\mathcal{G}) = L$

Vertical/horizontal arrows:

$\mathrm{Ver}(\mathcal{G}) = \mathrm{Hor}(\mathcal{G}) = \{e_A : A \to A : A \in L\}$,

- *e_A is the group unit of the Abelian group AP for each A in L.*
- *(Co)domains: $e_A \mathrm{dom} = e_A \mathrm{cod} = A$*
- *Composition: $e_A \circ e_A = e_A \bullet e_A = e_A$.*
- *Meets: $e_A \wedge e_B = A \wedge B$ to be that from L.*

Construction (cont'd)

Double cells: $\mathrm{Dbl}(\mathcal{G}) = \coprod_{A \in L} AP$

- *Disjoint union of all Abelian groups AP for A in L.*
- *A double cell a is contained in an Abelian group AP for some $A \in L$.*
- $a\mathrm{hdom} = a\mathrm{hcod} = a\mathrm{vdom} = \mathrm{vcod} = e_A$.
- *Composites: group products*
- *If $e_u \leq e_A = a\mathrm{hdom}$,*
  - *Restriction of a to e_u :*

$$(e_{u*}|a) = e_u *_u (a)\varphi_{u \leq A} = (a)\varphi_{u \leq A}$$

  - *Corestrictions are similarly defined.*

Notation

Denote the category of presheaves of Abelian groups on meet-semilattices by **AbMeetSLatt**.

Theorem

The categories **DIG** *and* **AbMeetSLatt** *are isomorphic.*

Recall:

- Kock showed double inverse semigroups are commutative.
- Double inverse semigroups are exactly presheaves of Abelian groups on meet-semilattices.

Theorem

Double inverse semigroups are commutative and improper. That is, $(S, \odot, \circledcirc)$ is a double inverse semigroup if and only if both $\odot$ and $\circledcirc$ are commutative inverse semigroup operations with $\circledcirc = \odot$.

Special thanks to Dr. Pronk, NSERC, FMCS and, of course, to each of you for listening!

. . . questions?