Faà di Bruno categories

R.A.G. Seely

John Abbott College
& McGill University

(Joint work with J.R.B. Cockett)

Francesco Faà di Bruno (1825-1888) was an Italian of noble birth, a soldier, a mathematician, and a priest. In 1988 he was beatified by Pope John Paul II for his charitable work teaching young women mathematics. As a mathematician he studied with Cauchy in Paris. He was a tall man with a solitary disposition who spoke seldom and, when teaching class, not always successfully. Perhaps his most significant mathematical contribution concerned the combinatorics of the higher-order chain rules. These results were the cornerstone of "combinatorial analysis": a subject which never really took off. It is the combinatorics underlying the higher-order chain rule which is of interest to us here.

Outline

- Cartesian differential categories
- The bundle fibration
- Faà di Bruno categories
- The comonad
- The coalgebras

Theorem *Cartesian differential categories are exactly standard coalgebras of the Faà di Bruno comonad.*

Key structure:

$$\frac{X \xrightarrow{f} Y \colon x \mapsto f(x)}{X \times X \xrightarrow{D(f)} Y \colon \langle a, s\rangle \mapsto \frac{\mathsf{d}f}{\mathsf{d}x}(s)\cdot a}$$

(linear in a but not in s)

Example:

If $f\colon \langle x, y, z\rangle \mapsto \langle x^2 + xyz, z^3 - xy\rangle$

then: $\frac{\mathsf{d}\langle x^2+xyz, z^3-xy\rangle}{\mathsf{d}\langle x,y,z\rangle} = \begin{pmatrix} 2x + yz & xz & xy \\ -y & -x & 3z^2 \end{pmatrix}$

and $\frac{\mathsf{d}\langle x^2+xyz, z^3-xy\rangle}{\mathsf{d}\langle x,y,z\rangle}(\langle r, s, t\rangle) = \begin{pmatrix} 2r + st & rt & rs \\ -s & -r & 3t^2 \end{pmatrix}$

and $\frac{\mathsf{d}\langle x^2+xyz, z^3-xy\rangle}{\mathsf{d}\langle x,y,z\rangle}(\langle r, s, t\rangle) \cdot \langle a, b, c\rangle =$

$$\langle (2r + st)a + rtb + rsc, -sa - rb + 3t^2c\rangle$$

Cartesian Differential Categories

1. Category $\mathbb{X}$, **Cartesian left additive**: hom-sets are commutative monoids & $f(g+h) = (fg) + (fh)$, $f0 = 0$.
   (h is **additive** if also $(f+g)h = (fh) + (gh)$ and $0h = 0$.)
   'Well-behaved' products: π_0, π_1, Δ additive
   f, g additive $\Rightarrow$ $f \times g$ additive.

2. Differential operator D:

$$\frac{X \xrightarrow{f} Y}{X \times X \xrightarrow[D[f]]{} Y}$$

(Ref: [Blute-Cockett-Seely] TAC 2009)

Eg (of "left additive"): the category of commutative monoids & **set** maps is left additive; the additive maps are homomorphisms.

Satisfying:

[CD.1] $D[f+g] = D[f] + D[g]$ and $D[0] = 0$

[CD.2] $\langle h+k, v\rangle D[f] = \langle h, v\rangle D[f] + \langle k, v\rangle D[f]$ and $\langle 0, v\rangle D_\times[f] = 0$

[CD.3] $D[1] = \pi_0$, $D[\pi_0] = \pi_0\pi_0$ and $D[\pi_1] = \pi_0\pi_1$

[CD.4] $D[\langle f, g\rangle] = \langle D[f], D[g]\rangle$

[CD.5] $D[fg] = \langle D[f], \pi_1 f\rangle D[g]$

[CD.6] $\langle\langle g, 0\rangle, \langle h, k\rangle\rangle D[D[f]] = \langle g, k\rangle D[f]$

[CD.7] $\langle\langle 0, h\rangle, \langle g, k\rangle\rangle D[D[f]] = \langle\langle 0, g\rangle, \langle h, k\rangle\rangle D[D[f]]$

[Dt.1] $\dfrac{\mathsf{d}(f_1+f_2)}{\mathsf{d}p}(s)\cdot a = \dfrac{\mathsf{d}f_1}{\mathsf{d}p}(s)\cdot a + \dfrac{\mathsf{d}f_2}{\mathsf{d}p}(s)\cdot a$ and $\dfrac{\mathsf{d}0}{\mathsf{d}p}(s)\cdot a = 0$;

[Dt.2] $\dfrac{\mathsf{d}f}{\mathsf{d}p}(s)\cdot(a_1+a_2) = \dfrac{\mathsf{d}f}{\mathsf{d}p}(s)\cdot a_1 + \dfrac{\mathsf{d}f}{\mathsf{d}p}(s)\cdot a_2$ and $\dfrac{\mathsf{d}f}{\mathsf{d}p}(s)\cdot 0 = 0$;

[Dt.3] $\dfrac{\mathsf{d}x}{\mathsf{d}x}(s)\cdot a = a$, $\dfrac{\mathsf{d}f}{\mathsf{d}(p,p')}(s,s')\cdot(a,0) = \dfrac{\mathsf{d}f[s'/p']}{\mathsf{d}p}(s)\cdot a$

and $\dfrac{\mathsf{d}f}{\mathsf{d}(p,p')}(s,s')\cdot(0,a') = \dfrac{\mathsf{d}f[s/p]}{\mathsf{d}p'}(s')\cdot a'$;

[Dt.4] $\dfrac{\mathsf{d}(f_1,f_2)}{\mathsf{d}p}(s)\cdot a = \left(\dfrac{\mathsf{d}f_1}{\mathsf{d}p}(s)\cdot a, \dfrac{\mathsf{d}f_1}{\mathsf{d}p}(s)\cdot a\right)$;

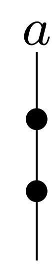
[Dt.5] $\dfrac{\mathsf{d}g[f/p']}{\mathsf{d}p}(s)\cdot a = \dfrac{\mathsf{d}g}{\mathsf{d}p'}(f[s/p])\cdot\left(\dfrac{\mathsf{d}f}{\mathsf{d}p}(s)\cdot a\right)$ (no variable of p may occur in f);

[Dt.6] $\dfrac{\mathsf{d}\frac{\mathsf{d}f}{\mathsf{d}p}(s)\cdot p'}{\mathsf{d}p'}(r)\cdot a = \dfrac{\mathsf{d}f}{\mathsf{d}p}(s)\cdot a$.

[Dt.7] $\dfrac{\mathsf{d}\frac{\mathsf{d}f}{\mathsf{d}p_1}(s_1)\cdot a_1}{\mathsf{d}p_2}(s_2)\cdot a_2 = \dfrac{\mathsf{d}\frac{\mathsf{d}f}{\mathsf{d}p_2}(s_2)\cdot a_2}{\mathsf{d}p_1}(s_1)\cdot a_1$

The Chain Rule

$$D[fg] = \langle D[f], \pi_1 f \rangle D[g]$$



$$\frac{\mathsf{d}g[f/x']}{\mathsf{d}x}(s) \cdot a = \frac{\mathsf{d}g}{\mathsf{d}x'}(f[s/x]) \cdot \left(\frac{\mathsf{d}f}{\mathsf{d}x}(s) \cdot a\right)$$

$$(fg)^{(1)}(s) \cdot a = g^{(1)}(f) \cdot (f^{(1)}(s) \cdot a)$$

The Bundle Fibration over $\mathbb{X}$

Objects: (A, X) (pairs of objects of $\mathbb{X}$)

Morphisms: $(f_*, f_1)\colon (A, X) \longrightarrow (B, Y)$: $f_*\colon X \longrightarrow Y$ in $\mathbb{X}$;
$f_1\colon A \times X \longrightarrow B$ in $\mathbb{X}$, additive in its first argument.

Composition: $(f_*, f_1)(g_*, g_1) = (f_* g_*, \langle f_1, \pi_1 f_* \rangle g_1)$
(Think $f_1 = D(f_*)$)

Additive structure: defined "component-wise"
$(A, X) \mapsto X$; $(f_*, f_1) \mapsto f_*$ is a fibration
If $\mathbb{X}$ is Cartesian left additive, so are the fibres, and so is the total category

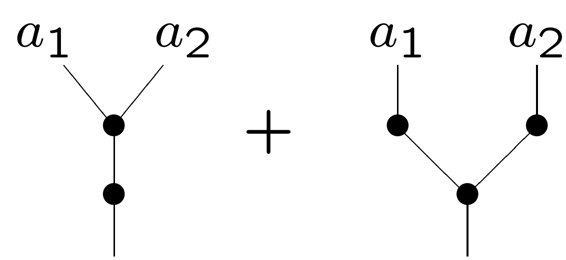
2nd Order Chain Rule

$$\frac{\mathsf{d}^{(2)} g(f(x))}{\mathsf{d}x}(s) \cdot a_1 \cdot a_2$$

$$= \frac{\mathsf{d}g}{\mathsf{d}x}(f(s)) \cdot \left(\frac{\mathsf{d}^{(2)} f}{\mathsf{d}x}(s) \cdot a_1 \cdot a_2\right)$$

$$+ \frac{\mathsf{d}^{(2)} g}{\mathsf{d}x}(f(s)) \cdot \left(\frac{\mathsf{d}f}{\mathsf{d}x}(s) \cdot a_1\right) \cdot \left(\frac{\mathsf{d}f}{\mathsf{d}x}(s) \cdot a_2\right)$$

i.e.

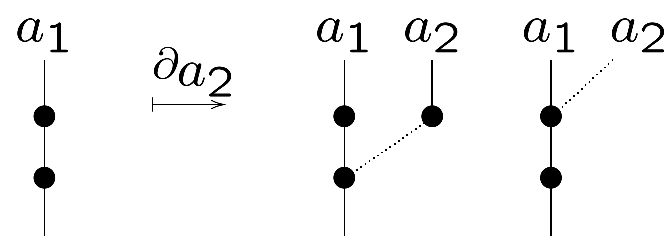


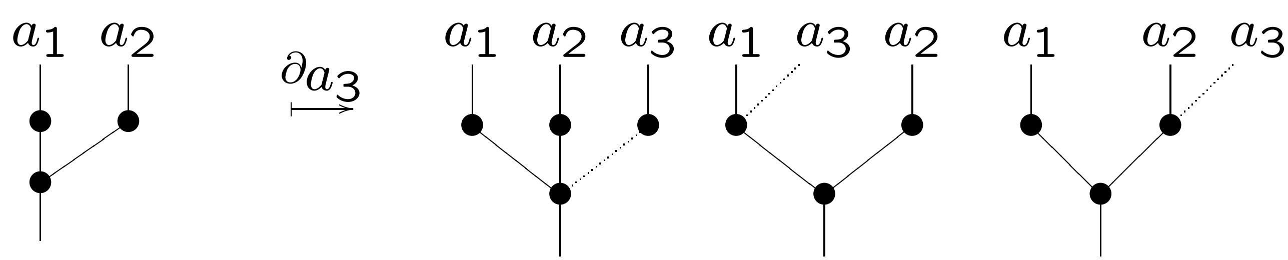
$$(fg)^{(2)}(s) \cdot a_1 \cdot a_2$$

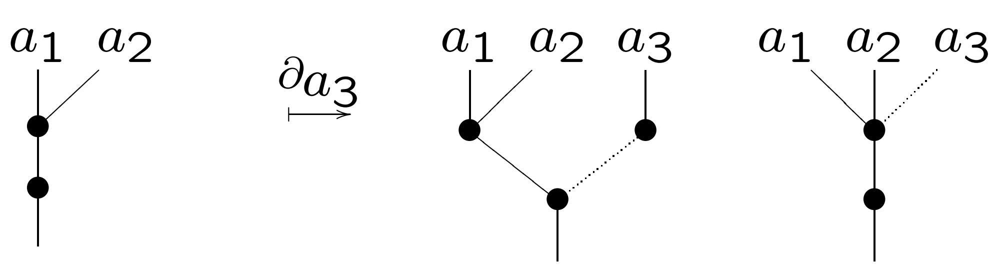
$$= g^{(1)}(f(s)) \cdot (f^{(2)}(s) \cdot a_1 \cdot a_2))$$

$$+ g^{(2)}(f(s)) \cdot (f^{(1)}(s) \cdot a_1) \cdot (f^{(1)}(s) \cdot a_2)$$

The differential of a symmetric tree







Faà($\mathbb{X}$), the Fàa di Bruno Fibration over $\mathbb{X}$

Objects: (A, X) (pairs of objects of $\mathbb{X}$)

Morphisms: $f = (f_*, f_1, f_2, \ldots)\colon (A, X) \longrightarrow (B, X)$, where:

$f_*\colon X \longrightarrow Y$ in $\mathbb{X}$;

for $r > 0$: $f_r\colon \underbrace{A \times \ldots \times A}_{r} \times X \longrightarrow B$ a "symmetric form" (*i.e.* additive and symmetric in the first r arguments

(think $f_r\colon A^{\otimes^r}/r! \times X \longrightarrow B$, even though $\mathbb{X}$ need not have $\otimes$)

Composition? This is where the higher order chain rules come in . . .

Faà di Bruno convolution

τ: a symmetric tree of height 2, width r, on variables $\{a_1, \ldots, a_r\}$;
$(A, X) \xrightarrow{f} (B, Y) \xrightarrow{g} (C, Z)$ in Faà($\mathbb{X}$).

Then $(f \star g)_\tau \colon \underbrace{A \times \ldots \times A}_{r} \times X \longrightarrow C$ is defined thus (for example): for τ the tree on the left, interpret it as the tree on the right:

$(f \star g)_\tau = g_2(f_*(x), f_1(a_3, x), f_3(a_1, a_2, a_4, x)) \colon A \times A \times A \times A \times X \longrightarrow C.$

NB: $(f \star g)_\tau$ is additive in each argument except the last whenever the components of f and g have this property.

$\iota_2^{a_1}$ is the (unique) height 2 width 1 tree (with variable a_1)

$\mathcal{T}_2^{a_1,\ldots,a_r} \; = \; \partial_{a_2,\ldots,a_r}(\iota_2^{a_1})$,

i.e. the bag of trees obtained by "deriving" $\iota_2^{a_1}$ r-times with respect to the given variables. (This is the set of *all* symmetric trees of height 2 and width r.)

The Faà di Bruno convolution (composition in Faà($\mathbb{X}$)) of f and g is given by setting $(fg)_* = f_* g_*$, and for $r > 0$

$$(fg)_r = (f \star g)_{\mathcal{T}_2^{\{a_1,\ldots,a_r\}}} = \sum_{n \,\cdot\, \tau \,\in\, \mathcal{T}_2^{a_1,\ldots,a_r}} n \cdot (f \star g)_\tau$$

(This is well-defined: permuting the variables of any $\tau \in \mathcal{T}_2^{a_1,\ldots,a_r}$ either leaves τ fixed or produces a new tree in $\mathcal{T}_2^{a_1,\ldots,a_r}$.)

Proposition *For any Cartesian left additive category* $\mathbb{X}$, Faà($\mathbb{X}$) *is a Cartesian left additive category.*

Faà: CLAdd $\longrightarrow$ CLAdd is a functor:

$$\mathbb{X} \mapsto \text{Faà}(\mathbb{X}) \ ; \ (f_*, f_1, \dots) \mapsto (F(f_*), F(f_1), \dots)$$

ϵ: Faà($\mathbb{X}$) $\longrightarrow \mathbb{X}$: $(A, X) \mapsto X, (f_*, f_1, \dots) \mapsto f$ is a fibration.
(and a natural transformation)

There is a functor (indeed, a natural transformation) δ: Faà($\mathbb{X}$) $\longrightarrow$ Faà(Faà($\mathbb{X}$)) so that (Faà, ϵ, δ) is a comonad on CLAdd.

On objects, $\delta: (A, X) \mapsto ((A, A), A, X)$

On morphisms, things are a bit "complicated". Some notation: we write $f = (f_*, f_1, f_2, \dots): (A, X) \longrightarrow (B, Y)$ as follows

$$f_*: X \longrightarrow Y \quad : \quad x \mapsto f_*(x)$$
$$f_n: A^n \times X \longrightarrow B \quad : \quad (a_{*1}, \dots, a_{*n}, x) \mapsto f_n(x) \cdot a_{*1} \cdot \ldots \cdot a_{*n}$$

We then define $\delta\colon \mathsf{Fa\grave{a}}(\mathbb{X}) \longrightarrow \mathsf{Fa\grave{a}}(\mathsf{Fa\grave{a}}(\mathbb{X}))$ as follows:

on objects, δ takes (A, X) to $((A, A), A, X)$.

On arrows, $f \mapsto \delta(f) = (f, f^{[1]}, f^{[2]}, \ldots)$ by setting

$$f_*^{[n]}\colon A^n \times X \longrightarrow B\colon (a_{*1}, \ldots, a_{*n}, x) \mapsto f_n(x) \cdot a_{*1} \cdot \ldots \cdot a_{*n}$$

$$f_r^{[n]}\colon (A^n \times A)^r \times (A^n \times X) \longrightarrow B\colon$$

$$\left(\begin{array}{c|c} a_{11} \ldots a_{1n} & a_{1*} \\ \vdots & \vdots \\ a_{r1} \ldots a_{rn} & a_{r*} \\ \hline a_{*1} \ldots a_{*n} & x \end{array}\right) \mapsto \sum_{\substack{s \leq n \,\&\, s \leq r \\ \&\, \mathsf{ramp}^s_{r,n}(\alpha \mid \gamma)}} f_{r+n-s}(x) \cdot a_{\alpha_1 1} \cdot \ldots \cdot a_{\alpha_n n} \cdot a_{\gamma_1 *} \cdot \ldots \cdot a_{\gamma_{r-s} *}$$

where the "ramp" condition amounts to choosing (for each $s \leq \min(r, n)$) s elements from $(a_{ij})_{i \leq r, j \leq n}$, at most one from each row and column, (this amounts to choosing a partial isomorphism) and constructing the function term as follows (for example,):

If σ is the following partial iso (here $n = 4$, $r = 5$, and $s = 3$):

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & a_{1*} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{2*} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{3*} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{4*} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{5*} \\ \hline a_{*1} & a_{*2} & a_{*3} & a_{*4} & x \end{array}\right) \quad \rightsquigarrow \quad \left(\begin{array}{cccc|c} \boxed{a_{11}} & a_{12} & a_{13} & a_{14} & a_{1*} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{2*} \\ a_{31} & a_{32} & a_{33} & \boxed{a_{34}} & a_{3*} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{4*} \\ a_{51} & \boxed{a_{52}} & a_{53} & a_{54} & a_{5*} \\ \hline a_{*1} & a_{*2} & a_{*3} & a_{*4} & x \end{array}\right)$$

Then construct

$$f^\sigma = f_6(x) \cdot a_{11} \cdot a_{52} \cdot a_{*3} \cdot a_{34} \cdot a_{2*} \cdot a_{4*}$$

$f_{\mathbf{6}}$ since we need $n + r - s = 6$ linear arguments. The linear arguments of f are determined by putting in the selected arguments and arguments from the bottom row and rightmost column corresponding to the rows and columns **not** containing a selected argument. Then we set $f_r^{[n]}$ to be the sum of all such expressions:

$$f_r^{[n]} = \sum_{\sigma \in \mathsf{ParIso}(r,n)} f^\sigma$$

Remark: The intended interpretation of $f_r^{[n]}$ is the r^{th} higher order differential term

$$\frac{\mathsf{d}^r f(x)\cdot a_1\cdot\cdots\cdot a_n}{\mathsf{d}(x,a_1,\ldots,a_n)}(x,a_1,\ldots,a_n)\cdot(a_1,a_{11},\ldots,a_{1n})\cdot\cdots\cdot(a_r,a_{r1},\ldots,a_{rn})$$

Properties: $f_r^{[n]}$ is additive, symmetric in its first r arguments.

$$(f+g)_r^{[n]} = f_r^{[n]} + g_r^{[n]}$$

If F is Cartesian left additive, $\mathsf{Faà}(F)(f^{[n]}) = (\mathsf{Faà}(F)(f))^{[n]}$

$\delta\colon \mathsf{Faà}(\mathbb{X}) \longrightarrow \mathsf{Faà}(\mathsf{Faà}(\mathbb{X}))$ is a functor, and is natural (as a natural transformation).

$(\mathsf{Faà}, \epsilon, \delta)$ is a comonad on CLAdd.

An example of the proofs:

Let's show that $\delta(f)\delta(g) = \delta(fg)$:

For the most part (as seen in the sequence of equations on the next slide) this involves expanding the definitions, followed by several applications of additivity; only the last step requires comment, as it involves a combinatorial argument.

$$
\begin{aligned}
\delta(f)\delta(g) &= \sum_{\tau_1,\tau_2} (\delta(f) \star \delta(g))_{\tau_1\times\tau_2} \\
&= \sum_{\tau_1,\tau_2} \left(\left(\sum_{\sigma:i\longrightarrow j} f^{\sigma} \right)_{ij} \star \left(\sum_{\sigma':k\longrightarrow l} g^{\sigma'} \right)_{kl} \right)_{\tau_1\times\tau_2} \\
&= \sum_{\tau_1,\tau_2} \left(\sum_{\sigma'} g^{\sigma'} \right) \left(\sum_{\sigma_{ij}:\alpha_i\longrightarrow\beta_j} f^{\sigma_{ij}} \right)_{ij} \\
&= \sum_{\tau_1,\tau_2} \sum_{\sigma'} g^{\sigma'} \left(\sum_{\sigma_{ij}} f^{\sigma_{ij}} \right)_{ij} \\
&= \sum_{\tau_1,\tau_2} \sum_{\sigma'} g^{\sigma'} \left(\sum_{\sigma_{ij}} f^{\sigma_{ij}} \right)_{ij\in\sigma'} \\
&= \sum_{\tau_1,\tau_2} \sum_{\sigma',\sigma_{ij},ij\in\sigma'} g^{\sigma}(\ldots, f^{\sigma_{ij}}, \ldots) \\
&= \sum_{\sigma:n\longrightarrow m} \sum_{\tau\in\mathcal{T}_{n+m-|\sigma|}} (f \star g)^{\sigma}_{\tau} \quad = \quad \delta(fg)
\end{aligned}
$$

The key combinatorial lemma is the equivalence of the following data:

- Partitions $\tau_1 = (\alpha_1, \ldots, \alpha_k), \tau_2 = (\beta_1, \ldots, \beta_l)$ and partial isomorphisms $\sigma' \colon k \longrightarrow l$ and $\sigma_{ij} \colon \alpha_i \longrightarrow \beta_j$ for $(i,j) \in \sigma'$

- Partial isomorphism $\sigma \colon n \longrightarrow m$ and partition of $n + m - |\sigma|$.

where n is the set partitioned by τ_1, m the set partitioned by τ_2, and σ is the union of the σ_{ij}.

We sketch the proof, with an example as illustration.

We shall frequently identify an integer n with the set of integers from 1 to n, unless otherwise stated. We shall represent a partial isomorphim by listing the pairs (i, j) where $i \mapsto j$.

Suppose we are given partitions $\tau_1 = (\alpha_1, \dots, \alpha_k), \tau_2 = (\beta_1, \dots, \beta_l)$ and partial isomorphisms $\sigma' \colon k \longrightarrow l$ and $\sigma_{ij} \colon \alpha_i \longrightarrow \beta_j$ for $(i, j) \in \sigma'$

Consider the following example:
$\tau_1 = ((1, 3), (2, 5), (4, 6))$
$\tau_2 = ((1, 2, 4), (3), (5))$ (so $k = l = 3$)
$\sigma' \colon 3 \longrightarrow 3 = \{(1, 3), (3, 1)\}$ (so *e.g.* $(2, 2)$ is not in σ)
$\sigma_{13} \colon \{1, 3\} \longrightarrow \{5\} = \{(3, 5)\}$
$\sigma_{31} \colon \{4, 6\} \longrightarrow \{1, 2, 4\} = \{(4, 4), (6, 1)\}$

Then $n = 6, m = 5, |\sigma| = 3$, and $\sigma \colon 6 \longrightarrow 5 = \{(3, 5), (4, 4), (6, 1)\}$

It remains to construct τ, a partition of an 8-element set.

From $\tau_1 = (\alpha_1, \ldots, \alpha_k), \tau_2 = (\beta_1, \ldots, \beta_l)$ we construct a set S with $n + m - |\sigma|$ elements, where n is the set partitioned by τ_1, m the set partitioned by τ_2, and σ is the union of the σ_{ij}.

S consists of pairs $(x, y) \in (n \cup \{*\}) \times (m \cup \{*\})$ as follows:
$(x, y) \in S$ if $(x, y) \in \sigma_{ij}$, $(i, j) \in \sigma'$.
$(x, *) \in S$ if $x \notin \pi_1\sigma$ ($\pi_1\sigma = 1^{\text{st}}$ components of elements of σ)
$(*, y) \in S$ if $y \notin \pi_2\sigma$ ($\pi_2\sigma = 2^{\text{nd}}$ components of elements of σ)

For our example, this gives
$S = \{(3, 5), (4, 4), 6, 1), (1, *), (2, *), (5, *), (*, 2), (*, 3)\}$.

We partition S as follows (write $a \sim b$ to mean a and b are in the same set of the partition S):

if $(x, y), (x', y') \in \sigma_{ij}$, then $(x, y) \sim (x', y')$
(this includes pairs containing $*$: if $x \notin \pi_1\sigma$, $x \in \alpha_i$ (*i.e.* x "comes from" σ_{ij}, but is not in its domain) then $(x, *)$ is in this same partition set, as is $(*, y)$ for $y \notin \pi_2\sigma$, $y \in \beta_j$ (*i.e.* y "comes from" σ_{ij} but is not in its codomain)

if $x, x' \in \alpha_i$ (so $x \sim x'$ in τ_1), then $(x, *) \sim (x', *)$

if $y, y' \in \beta_j$ (so $y \sim y'$ in τ_2), then $(*, y) \sim (*, y')$

In our example, this gives the 4-fold partition of S

$$\tau = (((4,4),(6,1),(*,2)),((3,5),(1,*)),((2,*),(5,*)),((*,3)))$$

(This completes one direction of the equivalence)

What's going on?

The given partitions and partial isos amount to this selection from a variable base:

$$\begin{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,4} \\ a_{3,1} & a_{3,2} & a_{3,4} \end{pmatrix} & \begin{pmatrix} a_{1,3} \\ a_{3,3} \end{pmatrix} & \boxed{\begin{pmatrix} a_{1,5} \\ \boxed{a_{3,5}} \end{pmatrix}} \\ \begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,4} \\ a_{5,1} & a_{5,2} & a_{5,4} \end{pmatrix} & \begin{pmatrix} a_{2,3} \\ a_{5,3} \end{pmatrix} & \begin{pmatrix} a_{2,5} \\ a_{5,5} \end{pmatrix} \\ \boxed{\begin{pmatrix} a_{4,1} & a_{4,2} & \boxed{a_{4,4}} \\ \boxed{a_{6,1}} & a_{6,2} & a_{6,4} \end{pmatrix}} & \begin{pmatrix} a_{4,3} \\ a_{6,3} \end{pmatrix} & \begin{pmatrix} a_{4,5} \\ a_{6,5} \end{pmatrix} \end{pmatrix}$$

and it's clear that what both sets of data are defining is the following term from the sums that define $\delta(f)\delta(g)$ and $\delta(fg)$:

$$g_4(x) \cdot (f_3(x) \cdot a_{44} \cdot a_{61} \cdot a_{*2}) \cdot (f_2(x) \cdot a_{35} \cdot a_{1*}) \cdot (f_2(x) \cdot a_{2*} \cdot a_{5*}) \cdot (f_1(x) \cdot a_{*3})$$

The other direction:
Suppose we are given a partial isomorphism $\sigma: n \longrightarrow m$ and a partition of $n + m - |\sigma|$.

We must construct partitions $\tau_1 = (\alpha_1, \ldots, \alpha_k), \tau_2 = (\beta_1, \ldots, \beta_l)$ and partial isomorphisms $\sigma': k \longrightarrow l$ and $\sigma_{ij}: \alpha_i \longrightarrow \beta_j$ for $(i,j) \in \sigma'$, of appropriate sizes.

Re-notate τ, so that it is a partition of the following set S, containing the pairs $(i,j) \in \sigma$, $(i,*)$ for $i \in n$ but $\notin \pi_1\sigma$, $(*,j)$ for $j \in m$ but $\notin \pi_2\sigma$.

Example: If $\sigma: 6 \longrightarrow 5 = \{(3,5),(4,4),(6,1)\}$, and
$\tau = ((1),(2,3),(4,5,8),(6,7))$, then

$S = \{(6,1)),(*,2),(*,3),(4,4),(3,5),(1,*),(2,*),(5,*)\}$ and
$\tau = (((6,1)),((*,2),(*,3)),((4,4),(3,5),(5,*)),((1,*),(2,*)))$

From the re-notated version of τ, it is easy to express $\tau = \tau_1 \times \tau_2$ as a product of partitions: consider the first components (ignoring $*$s) and the second components (ignoring $*$s). In our example, this gives

$\tau = ((6),(4,3,5),(1,2)) \times ((1),(2,3),(4,5))$
(so $k = l = 3$, and $n = 6$, $m = 5$ as required)

We can also construct (from τ) two partial isos, by ignoring the pairs with $*$s, and taking the remaining pairs from each partition. Note that by this construction, σ is the union of these partial isos, as required.

In our example, we get $\{(6,1)\}$ and $\{(4,4),(3,5)\}$, whose union is the $\sigma\colon 6 \longrightarrow 5 = \{(3,5),(4,4),(6,1)\}$ we started with.

Finally, we can construct $\sigma'\colon k \longrightarrow l$ by pairing the positions in τ_1 and τ_2 (equivalently the pairs in τ) which correspond to the partial isos above.

In our example this gives $\sigma' = \{(1,1),(2,3)\}$ (since $\{(6,1)\}$ assigns the first partition in τ_1 to the first partition in τ_2, and $\{(4,4),(3,5)\}$ assigns the second partition in τ_1 to the third partition in τ_2).

So $\sigma_{11} = \{(6,1)\}$ and $\sigma_{23} = \{(4,4),(3,5)\}$. And this completes the construction.

What's going on?

This time we have the following selection from the variable base:

$$
\begin{pmatrix}
\boxed{\begin{pmatrix} \boxed{a_{6,1}} \end{pmatrix}} & \begin{pmatrix} a_{6,2} & a_{6,3} \end{pmatrix} & \begin{pmatrix} a_{6,4} & a_{6,5} \end{pmatrix} \\
\begin{pmatrix} a_{4,1} \\ a_{3,1} \\ a_{5,1} \end{pmatrix} & \begin{pmatrix} a_{4,2} & a_{4,3} \\ a_{3,2} & a_{3,3} \\ a_{5,2} & a_{5,3} \end{pmatrix} & \boxed{\begin{pmatrix} \boxed{a_{4,4}} & a_{4,5} \\ a_{3,4} & \boxed{a_{3,5}} \\ a_{5,4} & a_{5,5} \end{pmatrix}} \\
\begin{pmatrix} a_{1,1} \\ a_{2,1} \end{pmatrix} & \begin{pmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{pmatrix} & \begin{pmatrix} a_{1,4} & a_{1,5} \\ a_{2,3} & a_{2,5} \end{pmatrix}
\end{pmatrix}
$$

and the common function term corresponding to this is
$g_4(x) \cdot (f_1(x) \cdot a_{61}) \cdot (f_2(x) \cdot a_{*2} \cdot a_{*3}) \cdot (f_3(x) \cdot a_{44} \cdot a_{35} \cdot a_{5*}) \cdot$
$(f_2(x) \cdot a_{1*} \cdot a_{2*})$

Coalgebras

Suppose $\mathbb{X}$, $D\colon \mathbb{X} \longrightarrow \text{Faà}(\mathbb{X})$ is a coalgebra (so $\epsilon D = 1$, $D\text{Faà}(D) = D\delta$). Since the bundle fibration is included in the Faà di Bruno fibration, we know (BCS, TAC2009) D induces a differential structure satisfying [CD.1]–[CD.5]. But [CD.6], [CD.7] ... ?

On objects: Let $D(X) = (D_0(X), D_1(X))$; then
$X = \varepsilon(D(X)) = \varepsilon(D_0(X), D_1(X)) = D_1(X)$ so $D_1(X) = X$.

Also

$$(D\text{Faà}(D))(X) = \text{Faà}(D)(D(X)) = \text{Faà}(D)(D_0(X), X) = ((D_0(D_0(X)), D_0(X))(D_0(X), X))$$

And

$$(D\delta)(X) = \delta(D_0(X), X) = ((D_0(X), D_0(X)), (D_0(X), X))$$

so $D_0(D_0(X)) = D_0(X)$, *i.e.* D_0 is an idempotent.

Call such a coalgebra in which D_0 is the identity on objects a **standard coalgebra**. Inside each coalgebra there always sits a standard coalgebra determined by the objects with $D_0(X) = X$.

On morphisms: Write $D(f) = (f, f^{(1)}, f^{(2)}, \ldots)$. The coalgebra equation for δ tells us these are equal:

$$\text{Faà}(D)(D(f)) = \begin{pmatrix} f & f^{(1)} & f^{(2)} & f^{(3)} & f^{(4)} & \ldots \\ f^{(1)} & (f^{(1)})^{(1)} & (f^{(2)})^{(1)} & (f^{(3)})^{(1)} & (f^{(4)})^{(1)} & \ldots \\ f^{(2)} & (f^{(1)})^{(2)} & (f^{(2)})^{(2)} & (f^{(3)})^{(2)} & (f^{(4)})^{(2)} & \ldots \\ f^{(3)} & (f^{(1)})^{(3)} & (f^{(2)})^{(3)} & (f^{(3)})^{(3)} & (f^{(4)})^{(3)} & \ldots \\ f^{(4)} & (f^{(1)})^{(4)} & (f^{(2)})^{(4)} & (f^{(3)})^{(4)} & (f^{(4)})^{(4)} & \ldots \\ \ldots \end{pmatrix}$$

$$\delta(D(f)) = \begin{pmatrix} f & D(f)_*^{[1]} & D(f)_*^{[2]} & D(f)_*^{[3]} & D(f)_*^{[4]} & \ldots \\ f^{(1)} & D(f)_1^{[1]} & D(f)_1^{[2]} & D(f)_1^{[3]} & D(f)_1^{[4]} & \ldots \\ f^{(2)} & D(f)_2^{[1]} & D(f)_2^{[2]} & D(f)_2^{[3]} & D(f)_2^{[4]} & \ldots \\ f^{(3)} & D(f)_3^{[1]} & D(f)_3^{[2]} & D(f)_3^{[3]} & D(f)_3^{[4]} & \ldots \\ f^{(4)} & D(f)_4^{[1]} & D(f)_4^{[2]} & D(f)_4^{[3]} & D(f)_4^{[4]} & \ldots \\ \ldots \end{pmatrix}$$

(which is enough to guarantee(!) [CD.6] & [CD.7])

(Why?)

Since $(f^{(1)})^{(1)} = D(f)_1^{[1]}$,

$$\left(\begin{array}{c|c} a_{1,1} & x_1 \\ a_{*,1} & x \end{array}\right) \mapsto (f^{(1)})^{(1)} \left(\begin{array}{c} x_1 \\ x \end{array}\right) \cdot \left(\begin{array}{c} a_{1,1} \\ a_{*,1} \end{array}\right)$$
$$= f^{(2)}(x) \cdot a_{*,1} \cdot x_1 + f^{(1)}(x) \cdot a_{1,1}$$

Setting $a_{*,1} = 0$ which yields **[CD.6]**:

$$(f^{(1)})^{(1)} \left(\begin{array}{c} x_1 \\ x \end{array}\right) \cdot \left(\begin{array}{c} a_{1,1} \\ 0 \end{array}\right) = f^{(1)}(x) \cdot a_{1,1}$$

and setting $a_{1,1} = 0$ yields **[CD.7]**:

$$\begin{aligned} & (f^{(1)})^{(1)} \left(\begin{array}{c} x_1 \\ x \end{array}\right) \cdot \left(\begin{array}{c} 0 \\ a_{*,1} \end{array}\right) \\ = \; & f^{(2)}(x) \cdot a_{*,1} \cdot x_1 \\ = \; & f^{(2)}(x) \cdot x_1 \cdot a_{*,1} \\ = \; & (f^{(1)})^{(1)} \left(\begin{array}{c} a_{*,1} \\ x \end{array}\right) \cdot \left(\begin{array}{c} 0 \\ x_1 \end{array}\right) \end{aligned}$$

So we have proved

Proposition *Every standard coalgebra of the Faà di Bruno comonad is a Cartesian differential category.*

To prove the converse involves some calculations using the term calculus of Cartesian differential categories. Here are some highlights.

Higher order derivatives

Define $\frac{\mathsf{d}^{(1)}t}{\mathsf{d}x}(s)\cdot a = \frac{\mathsf{d}t}{\mathsf{d}x}(s)\cdot a$ and

$$\frac{\mathsf{d}^{(n)}t}{\mathsf{d}x}(s)\cdot a_1\cdot\ldots\cdot a_n = \frac{\mathsf{d}\frac{\mathsf{d}^{(n-1)}t}{\mathsf{d}x}(x)\cdot a_1\cdot\ldots\cdot a_{n-1}}{\mathsf{d}x}(s)\cdot a_n$$

Then

$$\frac{\mathsf{d}t[x+s/y]}{\mathsf{d}x}(0)\cdot a = \frac{\mathsf{d}t}{\mathsf{d}y}(s)\cdot a \qquad (x \text{ not free in } s)$$

$$\frac{\mathsf{d}^{(2)}t}{\mathsf{d}x}(s)\cdot a_1\cdot a_2 = \frac{\mathsf{d}^{(2)}t}{\mathsf{d}x}(s)\cdot a_2\cdot a_1 \qquad (x \text{ not free in } a_1, a_2)$$

$$\frac{\mathsf{d}^{(n)}t}{\mathsf{d}x}(s)\cdot a_1\cdot\ldots\cdot a_n = \frac{\mathsf{d}^{(n)}t}{\mathsf{d}x}(s)\cdot a_{\sigma(1)}\cdot\ldots\cdot a_{\sigma(n)} \quad (\text{for any } \sigma\in\mathcal{S}_n.)$$

$$\frac{\mathsf{d}\frac{\mathsf{d}^{(n)}t}{\mathsf{d}z}(s)\cdot a_1\cdot\ldots\cdot x\cdot\ldots\cdot a_n}{\mathsf{d}x}(s')\cdot a_r = \frac{\mathsf{d}^{(n)}t}{\mathsf{d}z}(s)\cdot a_1\cdot\ldots\cdot a_r\cdot\ldots\cdot a_n$$

$$\frac{\mathsf{d}\frac{\mathsf{d}t}{\mathsf{d}x}(p)\cdot a}{\mathsf{d}y}(p')\cdot a' = \frac{\mathsf{d}^{(2)}t}{\mathsf{d}x}(p[p'/y])\cdot a[p'/y]\cdot\left(\frac{\mathsf{d}p}{\mathsf{d}y}(p')\cdot a'\right)$$
$$+\frac{\mathsf{d}t}{\mathsf{d}x}(p[p'/y])\cdot\left(\frac{\mathsf{d}a}{\mathsf{d}y}(p')\cdot a'\right) \qquad (\text{for } y\notin t)$$

Corollary: *In any cartesian differential category:*

$$\frac{\mathsf{d}^{(n)} g(f(x))}{\mathsf{d}x}(z) \cdot a_1 \cdot \ldots \cdot a_n = (f \star g)_{\mathcal{T}_2^{a_1,\ldots,a_n}}(z)$$

Furthermore

$$\frac{\mathsf{d}^{(m)} f_n(f_{n-1}(...(f(x))\cdots))}{\mathsf{d}x}(z) \cdot a_1 \cdots a_m = (f_1 \star f_2 \star \cdots \star f_n)_{\mathcal{T}_n^{a_1,\ldots,a_m}}(z)$$

In other words, the higher order derivatives connect with the Faà di Bruno convolution in exactly the right way, ...

... and so (after some technical calculations!):

Theorem *Cartesian differential categories are exactly standard coalgebras of the Faà di Bruno comonad.*