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Precise past—fuzzy future

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This paper examines the motivation and foundations of fuzzy sets theory, now some 20 years old, particularly possible misconceptions about possible operators and relations to probability theory. It presents a standard uncertainty logic (SUL) that subsumes standard propositional, fuzzy and probability logics, and shows how many key results may be derived within SUL without further constraints. These include resolutions of standard paradoxes such as those of the bald man and of the barber, decision rules used in pattern recognition and control, the derivation of numeric truth values from the axiomatic form of the SUL, and the derivation of operators such as the arithmetic mean. The addition of the constraint of truth-functionality to a SUL is shown to give fuzzy, or Lukasiewicz infinitely-valued, logic. The addition of the constraint of the law of the excluded middle to a SUL is shown to give probability, or modal S5, logic. An example is given of the use of the two logics in combination to give a possibility vector when modelling sequential behaviour with uncertain observations.

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Introduction

Time is a fundamental and fascinating what? It is tempting to say phenomenon, but perhaps in a Kantian sense it is a noumenon, something that is *prior* to reality, or, idealistically, imposed by our minds in structuring experience. In the context of the title of this presentation it suffices to note that time presents many unsolved problems and its nature is still not understood (Gold, 1967). One of the mysteries of time is the asymmetry between past and future which contrasts with the symmetrical form of most physical laws—it corresponds to the past somehow having *happened* and thus being fully determined and certain, whereas the future is a vague path through possible worlds, indeterminate and uncertain. We do, indeed, think of there being a *precise past* and a *fuzzy future.*

In this presentation I shall capitalize upon the play on words possible with the title, taking as an overall theme the transition that is taking place in science and engineering from the extreme requirement for *precision* of past scientific paradigms to the balanced position of accurately representing the intrinsic imprecision of a *possibilistic* universe in future scientific paradigms. However, I shall also return to the modelling of a world experienced sequentially through time, and will also seek to rectify some of the historical mis-statements of fuzzy sets theory that prove that the past is not so certain as we suppose.

Origins of fuzzy sets theory

I remember being fascinated 10 years ago by a paper on *A trilogy of errors in the history of computing* presented at the Joint U.S.A.–Japan Computing Conference

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(Metropolis & Worlton, 1972). For major errors to arise in recording the facts of a subject area then only some 25 years old, and for these to be propagated, and hence highly cross-confirmed, in a wide range of textbooks is astonishing. Fuzzy sets theory is this year (1982) 20 years old, and already some of what we take to be well established truths are suspect. In this presentation I will concentrate on some misconceptions about fuzzy sets theory that seem significantly misleading to me. In particular I shall examine the links between fuzzy sets theory and probability theory and show that these are closer and richer in significance than is often stated.

Perhaps the first "error" made in our literature references is to assume that Zadeh's (1965) paper is the first reference to fuzzy sets. It is indeed the first mathematical presentation, but the motivation for the concept, and the term *fuzzy*, predate this by three years (Zadeh, 1962):

> There are some who feel . . . the fundamental inadequacy of the conventional mathematics—the mathematics of precisely-defined points, functions, sets, probability measures, etc.—for coping with the analysis of biological systems, and that to deal effectively with such systems, we need a radically different kind of mathematics, the mathematics of fuzzy or cloudy quantities which are not describable in terms of probability distributions. Indeed the need for such mathematics is becoming increasingly apparent even in the realm of inanimate systems.

It places the terminology in its proper context to note that today we might be at a conference of the *North American Cloudy Information Processing* group, and have a journal of *Cloudy Sets and Systems!*

Zadeh's (1962) paper was entitled *From circuit theory to system theory* and this is a very significant context for motivating the development of fuzzy sets. In electronic circuits and their applications to computing, communications and control, we find the apex of modern scientific achievement and our greatest technological triumphs. We might also believe circuit theory to be a confirmation of the intrinsic value of the precisiation process in the paradigm of science—circuits do behave with precision and do follow the underlying mathematics with uncanny veracity—we can design in theory and then implement with exactitude. However, it would be more correct to say that circuits can be *made* to behave with precision—electronics is our ultimate artefact designed to enable us to use our scientific and mathematical techniques, not working *because* they are right, but made to work *as if* they are right. It is when we are fooled by the success of our paradigm in predicting and controlling our artefacts into believing that it is also a tool for predicting and controlling the natural world that problems occur. The notions of *state*, *stability*, *adaptivity*, and so on, that had served us so well in engineering, when transferred to biological, social and economic systems became themselves suspect.

Stability is an intrinsically imprecise concept and when precisely analysed explodes into a richness of definitions necessary to match the variety of the world but far removed from our intuitive concept of a stable system (Habets & Pfeifer, 1973). The notion of a *state*, so clear in system design, becomes a mathematical artefact when it has to be inferred from system behaviour (Zadeh, 1964). The notion of *adaptivity* is particularly interesting because it has biological roots and yet has played an important role in circuit and control theory (Zadeh, 1963)—again the notion explodes with combinatorial complexity when analysed precisely (Gaines, 1972). Zadeh's (1962) paper marked a turning point in his own thought processes—from a major involvement during the 1950s with the frontiers of mathematical system theory—culminating in the early1960s in formal definitions of basic systemic concepts—calling in 1962 for new foundations for these concepts when applied not only to biological but also to inanimate systems—and providing it in 1965 with fuzzy sets theory. To take the new tools we now have and go back to the fundamental system concepts of state, stability, adaptivity, and so on, and give them exact definitions that accurately reflect their intrinsic imprecision—that is still a task for the fuzzy future.

Links with nonstandard logics

Fuzzy sets theory has very close links with the nonstandard multivalued logics of Lukasiewicz (Rescher, 1969), but it has gone beyond this in two distinct directions: first, in developing a linguistic semantics for the logic in terms of *fuzzy hedges* (Zadeh, 1973), and secondly in its wealth of practical applications. I safely predict now that in 25 years time textbook writers may well account for fuzzy sets theory as the development of semantics for Lukasiewicz logic that gave it a practical importance previously only possessed by the standard predicate calculus. It did not actually happen that way but history is itself subject to revision through rationalization.

Nonstandard logics arise when we reject the binary, black-and-white, distinctions that we generally presuppose—that our terms of reference, our data, our measurements, our plans, and so on, are well defined. In logic this occurs classically as the problem of the *borderline case*, the rose that is neither red nor not-red but somewhere in the excluded middle between the two (Sanford, 1975). It is the basis of the *antinomy of predication* that Russell discovered in Frege's foundations of arithmetic at a time when they seemed complete (Kneale & Kneale, 1962): that every predicate, every distinction, does not serve to define a set—the axiom of comprehension that assumes this unrestricted predication leads to paradoxes in set theory. Zadeh's choice of Lukasiewicz logic as a foundation for his theory of fuzzy systems was vindicated in its fundamentals 10 years later by the work of Maydole (1975) and White (1979): the former showing that most nonstandard logics also give rise to paradoxes in set theory, and the latter that Lukasiewicz logic does not.

The significance of allowing imprecise distinctions in system theory may be seen at a fundamental level by noting that a system is nothing more nor less than a distinction. By making a distinction we cut out part of a world and give it additional systemic characteristics just by its being distinguished (Gaines, 1981*b*). This concept has been stated most clearly by Spencer Brown in his *Laws of Form* (Brown, 1969, p. v):

> The theme of this book is that a universe comes into being when a space is severed or taken apart. . . . By tracing the way we present such a severance, we can begin to reconstruct, with an accuracy and coverage that appear almost uncanny, the basic forms underlying linguistic, mathematical, physical and biological science, and can begin to see how the familiar laws of our own experience follow inexorably from the original act of severance.

Brown goes on to develop a "calculus of distinctions" which may be interpreted as standard propositional calculus (Schwartz, 1981), but which also subsumes a variety of nonstandard logics through other interpretations (Kohout & Pinkava, 1980); these include Lukasiewicz/fuzzy logic as one of the forms.

Nonstandard logics arise when we accept the uncertainty of our distinctions and note that definiteness in distinction is not available to us. Thus, they give us a calculus of systems that can encompass the essential imprecision of our real world data without forcing us to introduce artefacts in order to satisfy a meaningless requirement for precision.

Classical, probabilistic and fuzzy logics

The applications of fuzzy reasoning in system studies now encompass a very wide range of disciplines from quantum physics and biology to computer and management sciences (Gaines & Kohout, 1977). It has also proved its practical worth in the solution of engineering control problems previously regarded as intractable (Mamdani & Assilian, 1975; Mamdani & Gaines, 1981). These applications perhaps also indicate a puzzle in that we already had techniques to deal with uncertainty. What is new, and how does it relate to the old? In the early years of fuzzy set theory it was necessary to emphasize the new, to draw attention to the differences and their value, and this led to an overemphasis in the literature on the lack of relation of fuzzy set theory to previous logics of uncertainty, such as probability theory. However, there are both fundamental differences and fundamental similarities and relationships.

The following technical sections give a rigorous formulation of these relationships, bringing out the similarities and differences, and attempting to place these in the context of their significance in systems analysis. In particular, it is shown that both probability and fuzzy logics may be derived from a weaker logic that is axiomatized as Lukasiewicz infinitely valued logic but is not truth-functional. Probability logic has the added axiom of the law of the excluded middle, and fuzzy logic has the added axiom of truth functionality. However, key results for systems analysis may be derived with the weaker logic and hence apply to both probability and fuzzy logics. Some *paradoxes* of classical logic are used to derive these results, including the generation of numerical truth values, probabilities, degrees of membership, systems rules and arithmetic means, as features of all the logics considered. It remains to undertake the mammoth task of rebuilding system theory on logical foundations that are difficult for us who are over-tutored in the standard results and import them too readily when they are not legitimate (Ackermann, 1967).

A standard uncertainty logic as a valuation on a lattice

If the links between classical logic, probability and fuzzy logic are to appear clearly, so that similarities and differences are transparently obvious, it is useful to develop all three through parallel paths until the essential divergencies are apparent. We here develop a logical system through a valuation on a standard lattice of propositions which corresponds to the classical propositional calculus if the valuation is binary, but to both probability and fuzzy logics if it is multivalued in the unit interval. The final divergence between probability and fuzzy logics is made through the addition of differing simple and natural requirements which the logic might additionally meet. However, it is noted that most of the key features and theorems of both logics are available before this divergence—*features and theorems significant for many applications are independent of the particular logic used.*

Let $L(X, F, T, \vee, \wedge)$ be the free lattice generated by a set of elements, X, under the two (idempotent, commutative) semigroup operations, $\vee$, $\wedge$, with maximal element

T and minimal element F, i.e. L satisfies

(P1) $\forall x \in L$, $x \vee x = x \wedge x = x$,
(P2) $\forall x, y \in L$, $x \vee y = y \vee x$, $x \wedge y = y \wedge x$,
(P3) $\forall x, y, z \in L$, $x \vee (y \vee z) = (x \vee y) \vee z$, $x \wedge (y \wedge z) = (x \wedge y) \wedge z$,
(P4) $\forall x, y \in L$, $x \vee (x \wedge y) = x$, $x \wedge (x \vee y) = x$,
(P5) $\forall x \in L$, $x \vee T = T$, $x \wedge T = x$, $x \vee F = x$, $x \wedge F = F$

the idempotent, commutative, associative and adsorption postulates, together with a definition of the minimal and maximal elements (Birkhoff, 1948, p.18). The usual order relation may also be defined:

(P6) $\forall x, y \in L, \quad x \leq y \leftrightarrow \exists z \in L: y = x \vee z.$

Now suppose that every element of L is assigned a "truth-value", such as a "probability" or "degree of membership", in the closed interval [0,1] by a continuous order-preserving function $p: L \to [0,1]$ subject to:

(P7) $p(F) = 0, \quad p(T) = 1,$
(P8) $\forall x, y \in L, \quad x \leq y \to p(x) \leq p(y),$
(P9) $\forall x, y \in L, \quad p(x \vee y) + p(x \wedge y) = p(x) + p(y),$

i.e. p is a continuous, order-preserving *valuation* on L (Birkhoff, 1948, p. 74). Note that for p to exist the lattice must be modular, and that we have

$$\forall x, y \in L, \quad p(x \wedge y) \leq \min(p(x), p(y)) \leq \max(p(x), p(y)) \leq p(x \vee y), \tag{1}$$

Now the relation defined by

(P10) $\forall x, y \in L \quad x \equiv y \leftrightarrow p(x \vee y) = p(x \wedge y)$

is a *congruence* on L (Birkhoff, 1948, p. 77) so that

$$\forall x, y, z \in L, \quad x \equiv y \to (x \vee z) = (y \vee z), \quad (x \wedge z) = (y \wedge z), \tag{2}$$

which in its terms implies

$$\forall x, y, z \in L, \quad p(x \vee y) = p(x \wedge y) \to p(x \vee z) = p(y \vee z), \quad p(x \wedge z) = p(y \wedge z), \tag{3}$$

i.e. $x \equiv y$ means that y may be substituted freely for x in both L and p expressions without changing their value. Thus with respect to the valuation p the relation $\equiv$ is one of *logical equivalence.*

Note that $x \equiv y$ is a relation on L, not an element in L. However, it is useful to consider the possibility of there existing within L an element z that *behaves* as $x \equiv y$ in that

$$z = T \leftrightarrow x \equiv y. \tag{4}$$

We shall regard the symbol $x \equiv y$ as also standing for the class of such elements, if any exist. We can then write relation (4) as

$$\forall x, y \in L, \quad p(x \equiv y) = 1 \leftrightarrow x \equiv y. \tag{5}$$

Note that the use of the expression $p(x \equiv y)$ does not mean that all members of the equivalence class have the same valuation. They need only have the same valuation when this is 1 and then the constraint expressed in relation (5) applies. This is a common feature of multivalued logic, that functions such as equivalence are constrained to have the obvious semantics when they express the truth of a relation but otherwise can take a variety of forms giving essentially different logics (Rescher, 1969).

In terms of structure some of the features of both probability and fuzzy logic are already apparent. (P8) and (P9) together ensure that the logic gives the standard truth tables for conjunction and disjunction when restricted to binary values. (P9) is a property we associate with additivity of probability and inequality (1) brings in the max/min functions we associate with fuzzy logic—both features belong to the system defined so far.

Implication in a SUL as a distance function

We can complete this system by defining an implication function in terms of a natural distance on the lattice L. Define

(P11) $\forall x, y \in \mathrm{L}, \quad d(x, y) = p(x \wedge y) - p(x \vee y).$

The function d is a quasimetric on L (Birkhoff, 1948, p. 77) satisfying

$$\forall x \in \mathrm{L}, \qquad d(x, x) = 0 \tag{6}$$
$$\forall x, y \in \mathrm{L}, \qquad 0 \le d(x, y) \le 1, \tag{7}$$
$$\forall x, y, z \in \mathrm{L}, \qquad d(x, z) \le d(x, y) + d(x, z). \tag{8}$$

The congruence of (P10) is such that the quotient lattice of L under it has d as a true metric such that

$$\forall x, y \in \mathrm{L}, \qquad d(x, y) = 0 \leftrightarrow x \equiv y. \tag{9}$$

Hence, consistent with constraint of relation (5), we can define a measure of the equivalence between two elements in terms of the distance between them:

(P12) $\forall x, y \in \mathrm{L}, \quad p(x \equiv y) = 1 - d(x, y) = 1 - p(x \vee y) + p(x \wedge y).$

This gives all members of the equivalence class denoted by $x \equiv y$ the same valuation.

It is then natural to define a variation of the extent to which x "implies" y by noting that if $x \supset y$ is true in standard logic then $x \wedge y = x$ and $x \vee y = y$, and letting the degree of equivalence of what would normally be equalities define

(P13) $\forall x, y \in \mathrm{L}, \quad p(x \supset y) = p(x \equiv x \wedge y) = 1 - d(x, x \wedge y) = 1 - p(x) + p(x \wedge y)$
$= 1 + p(y) - p(x \vee y) = 1 - d(y, x \vee y) = p(x \equiv x \vee y).$

Again $x \supset y$ should be read as a symbol for a set of lattice elements, possibly empty, such that they satisfy the constraint of (P13).

Negation may be defined in the usual way (Prior, 1962, p. 50) in terms of equivalence, and (P5), (P7) and (P13) may be used to obtained a well-defined value for $p(\bar{x})$, the valuation of each member of the equivalence class $\bar{x}$, in terms of $p(x)$:

(P14) $\forall x \in \mathrm{L}, \quad p(\bar{x}) = p(x \equiv \mathrm{F}) = p(x \supset F) = 1 - p(x),$

and we have, conversely,

$$\forall x \in \mathrm{L}, \qquad p(x \equiv \mathrm{T}) = p(\mathrm{T} \supset x) = p(x). \tag{10}$$

The logical system we have defined through a valuation over a lattice of propositions is a simple and natural one that will be shown to include both probability and fuzzy logics as special cases. It has so many of the expected features and theorems of both probability and fuzzy logics that I have previously termed it a *standard uncertainty logic* (SUL) (Gaines, 1978*a*). SUL defaults as expected in the binary case in that the definitions of valuations for conjunction, disjunction, equivalence, implication and negation have the standard truth tables when the valuation can only take the values 0 and 1. It does not have the law of the excluded middle as a theorem. However, if element exist within L that correspond to $x \equiv y$, $x \supset y$, and $\bar{x}$, then we have from (P9) and (P13) that

$$\forall x, y \in \mathrm{L}, \qquad p(x \vee (x \supset y)) + p(x \wedge (x \supset y)) = 1 + p(x \wedge y), \tag{11}$$

so that, substituting F for y:

$$\forall x \in \mathrm{L}, \qquad p(x \vee \bar{x}) + p(x \wedge \bar{x}) - 1, \tag{12}$$

showing that the *law of the excluded middle* $(p(x \vee \bar{x}) = 1)$ and the *law of contradiction* $(p(x \wedge \bar{x}) = 0)$ are equivalent in the logical system defined in that postulating one allows the other to be derived.

Some system-theoretic results in a SUL

Certain practical results hold for a standard uncertainty logic without any further postulates that are key to a range of system-theoretic applications to decision and control. The implication valuation of (P13) has the property that

$$\forall x, y \in \mathrm{L}, \qquad p(y) = p(x \vee y) - 1 + p(x \supset y) \geq p(x) - (1 - p(x \supset y)), \tag{13}$$

which enables a lower bound to be placed on the truth value of y given those for x and $x \supset y$, thus allowing a multivalued form of *modus ponens*.

This result is particularly interesting if applied to the problem of paradoxical results generated by long chains of inference. The classic paradox is that of deciding whether someone is bald: clearly someone with no hair is bald, and surely someone who differs from a bald man by having only one additional hair can also be considered bald, but then by repeated extrapolation someone with any amount of hair is bald. Classical logic has difficulty with this type of argument because it makes no provision for an inference to be almost, but not quite, valid: either $x \supset y$ is true or it is false. If we read x as $\mathrm{B}(n-1)$: a man with $n-1$ hairs is bald, and y as $\mathrm{B}(n)$: a man with n hairs is bald, then the paradox arises because $x \supset y$ is so nearly valid that we do not wish to deny it, yet when repeatedly applied it leads to unacceptable inference, that is, from the truth of $\mathrm{B}(0)$ we can infer the truth of $\mathrm{B}(n)$ for any n.

SUL, however, allows for an inference rule to be nearly, but not quite valid, for example we may set

$$\mathrm{p}(x \supset y) = 1 - \alpha \quad \text{or} \quad p(\mathrm{B}(n-1) \supset \mathrm{B}(n)) = 1 - \alpha, \tag{14}$$

where α is very small, so that the truth value of $\mathrm{B}(n-1) \supset \mathrm{B}(n)$ is very near to, but not quite, 1. Then relation (13) gives us

$$p(\mathrm{B}(n)) \geq p(\mathrm{B}(n-1)) - \alpha, \tag{15}$$

from which we may infer, by repeated application that

$$p(\mathrm{B}(n)) \geq p(\mathrm{B}(0)) - n \times \alpha, \tag{16}$$

so that, even though B(0) might be true, and the inference rule $\mathrm{B}(n-1) \supset \mathrm{B}(n)$ is nearly true, the truth value of $\mathrm{B}(n)$ declines with the length of the chain of inference necessary to derive it. This is an intuitively satisfying model for this type of paradox and its resolution.

When we are actually dealing with rules of inference that are valid in that $p(x \supset y) = 1$, we can derive from relation (13)

$$\forall x, y \in \mathrm{L} \qquad p(x \supset y) = 1 \rightarrow p(y) \geq p(x), \tag{17}$$

so that the form of implication defined for a SUL satisfies the requirement (Lee & Chang, 1971) that the assertion of $x \supset y$ may be used to infer that $p(y) \geq p(x)$. The result of relation (17) generalizes to a very significant inequality that, when y is constrained by rules of the form $x \supset y$ ("if x then y"), a lower bound of $\max(p(x))$ may be placed on $p(y)$:

$$\forall x, y \in \mathrm{L} \text{ such that } x \supset y, \qquad p(y) \geq \max(p(x)). \tag{18}$$

This is a key pattern of inference in the decision and control applications of fuzzy reasoning (Mamdani & Assilian, 1975) where a number of "rules" are given that are then used to derive actions from actual data. It is worthwhile emphasizing that this result, although generally thought of as one of "fuzzy reasoning" actually applies to "probabilistic reasoning" also, that is, if we have a *rule* that from knowing x we may derive y, then if we know the probability that x has occurred we may infer that the probability that y has occurred is at least as great as this. Gaines (1975*a*) showed that the control strategy derived by Mamdani & Assilian (1975) through the application of rules of fuzzy logic to their data was replicated when probabilistic rules were applied to the same data. Thus, relation (18) expresses a common pattern of inference in both probabilistic and fuzzy control and decision making.

Derivation of fuzzy and probability logics from a SUL

The unusual feature of SUL is that it is *non truth-functional* in that the values of $p(x \vee y)$, $p(x \wedge y)$, $p(x \equiv y)$ and $p(x \supset y)$, cannot be obtained from $p(x)$ and $p(y)$. However, there is only one degree of freedom since (P9) ensures that the fixing of $p(x \wedge y)$ also fixes $p(x \vee y)$ and vice versa, and even this freedom is restricted by the inequality of (1). If we require the SUL to be strongly truth-functional such that the valuation of any connective can be determined in terms of the valuations of its constituents, then the arguments of Bellman & Giertz (1973) show that the outer inequalities of (1) become equalities and the logic is Lukasiewicz' infinitely valued logic (Rescher, 1969, section 6). This has the connectives used by Zadeh (1965) for fuzzy logic:

(L1) $\quad 0 \leq p(x) \leq 1,$

(L2) $\quad p(\bar{x}) = 1 - p(x),$

(L3) $\quad p(x \wedge y) = \min(p(x), p(y)),$

(L4) $\quad p(x \vee y) = \max(p(x), p(y)),$

(L5) $p(x \supset y) = \min(1, 1 - p(x) + p(y))$,
(L6) $p(x \equiv y) = \min(1 - p(x) + p(y), 1 + p(x) - p(y))$.

This is a significant result because it explains some of the attraction of fuzzy logic. The requirement of truth functionality is one which makes the logic computationally tractable, both for people and for computers. Whether the logic is actually truth functional in a particular application clearly depends on the semantics of that application, but the assumption that leads to the connectives of fuzzy logic is a way of resolving the uncertainty about derived truth values in a consistent fashion. Any other resolution leaves the valuation of a compound proposition dependent not just upon the valuation of its constituents but also upon their structure. *With any other assumption we have to remember not only the truth values of derived propositions but also the way in which they were derived.*

This truth functionality does not hold for probability logic and this may be derived from SUL by an alternative additional assumption, that the law of contraction holds so that $p(x \vee \bar{x}) = 1$. Equation (11) shows that this is equivalent to assuming that the law of the excluded middle holds, and either assumption gives a standard probability logic (Rescher, 1969, p. 185):

(R1) $0 \leq p(x)$,
(R2) $p(x \vee y) = p(x) + p(y)$ if x and y are mutually exclusive,
(R3) $p(x) = p(y)$ if x and y are logically equivalent,
(R4) $p(x \vee \bar{x}) = 1$.

which is also the system defined by Fenstad (1967) to represent probabilities on a first order language.

Thus an SUL system of valuations on a lattice as defined above subsumes both probability and fuzzy logics and enables key inference rules common to both to be derived. The logics only diverge in a final step whereby the use of fuzzy logic assumes strong functionality and the use of probability logic assumes the law of the excluded middle. I emphasize this not to suggest that the differences between the two logics is trivial, but rather to show that it does not necessarily lie in major part in the connectives used. For many applications an SUL is adequate and this encompasses both forms of connective. It is rather to the forms of the applications themselves that we should look for the interesting differences between the fuzzy and probability logics.

Axiomatic form of SUL

There is an alternative approach to the SUL and its relation to fuzzy and probability logics that brings out further features of the relationship. In particular it shows how both logics relate to the analysis of *possibilistic* systems (Gaines & Kohout, 1975; Zadeh, 1978). For this approach we go back to the formal axiomatic systems underlying the logics.

Lukasiewicz infinitely-valued logic has the underlying formal system (Rescher, 1969; Gaines, 1976) of the following four axioms in terms of falsity and implication as primitives:

(A1) $\forall x, \quad F \supset x$,

that from the assertion of falsity, or a contradiction, any proposition may be inferred;

(A2) $\forall x, y, \quad x \supset (y \supset x)$,

that a true proposition may be inferred from any proposition, the so-called paradox of material implication;

(A3) $\forall x, y, \quad ((x \supset y) \supset y)) \supset ((y \supset x) \supset x)$,

that disjunction and conjunction are symmetric (see DD and DC, following);

(A4) $\forall x, y, z, \quad (x \supset y) \supset ((y \supset z) \supset (x \supset z) \supset x)$,

that implication is transitive.

To these must be added the inference rules of substitution, that any well-formed formula may be substituted uniformly for any variable in a theorem, and *modus ponens*

(MP) $\forall x, y, \quad x, x \supset y \rightarrow y$,

that from x and $x \supset y$ we may infer y.

The remaining logical connectives may then be defined as

(DN) Negation: $\bar{x}$ for $x \supset F$,
(DD) Disjunction: $x \vee y$ for $(x \supset y) \supset y$,
(DC) Conjunction: $x \wedge y$ for $\sim(\sim x \vee \sim y)$,
(DE) Equivalence: $x \equiv y$ for $(x \supset y) \wedge (y \supset x)$.

This logical system reduces to the standard propositional calculus (PC) if the additional axiom is assumed that

(A5) $\forall x, y, \quad ((x \supset y) \supset x) \supset x$,

which from (DC) is equivalent to

(A5′) $\forall x, y, \quad (x \supset y) \vee x$,

which, if we substitute F for y, gives us

$\forall x, \quad (x \supset F) \vee x$ or $\bar{x} \vee x$,

which is the law of the excluded middle.

The axioms (A1)–(A4), inference rules, and definitions are satisfied by a SUL and serve to define it in non-numeric terms. They are generally thought of as axioms for Lukasiewicz logic and Wajsberg (1967) derives the numeric form given in (L1)–(L6) from them. However, he assumes that the logic is truth-functional and hence his result is parallel to that of the previous section—that an SUL that is truth-functional is precisely Lukasiewicz' infinitely-valued logic.

The other parallel result is that a probability logic as defined by (R1)–(R4) also obeys these axioms together with (A5) also—that an SUL with the law of the excluded middle is precisely Rescher's probability logic. Rescher (1963) has shown that probability logic is formally equivalent to the modal logic S5, and this is apparent in axioms (A1)–(A5) being those of the implicational fragment of S5 provided $\supset$ is regarded as a "strict" implication (Lemmon, Meredith, Meredith, Prior & Thomas, 1969; Anderson & Belnap, 1975).

It is interesting to summarize these results by relating them back to some of the proposed applications of the logical calculi also. Gaines & Kohout (1975) and Zadeh (1978) have proposed that Lukasiewicz or fuzzy logic be regarded as a basis for a *possibilistic* system theory and given illustrations of its applications to database systems (Gaines, 1981*a*) and to the extension of probabilistic sequential system identification (Gaines, 1977) to uncertain sequential data (Gaines, 1979). The modal logic S5 is the classical logic of possibility of Lewis & Langford (1932). It is also the system that

Rescher (1963) has shown to underly probability logic which is itself used in practice as a basis for possibilistic system theory. The standard uncertainty logic developed in this section draws all these systems together and shows both their close inter-relationships and also their essential differences; Fig. 1 illustrates this in diagrammatic form.

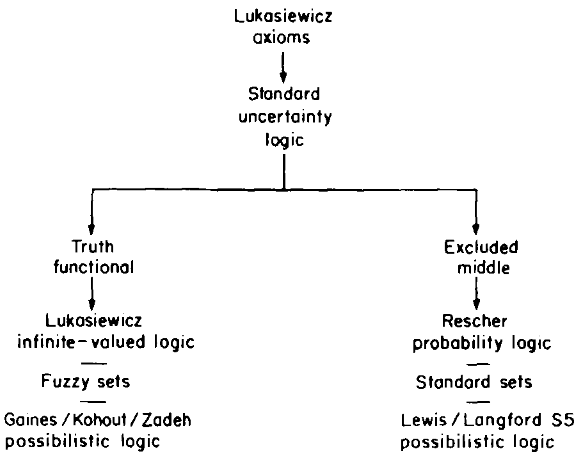

FIG. 1. Relations between SUL and possibilistic logics.

The origin of numeric truth values

One feature of both fuzzy and probabilistic logics, as models of uncertain reasoning, that is often taken as counter-intuitive is their use of numbers to represent probabilities or degrees of membership, and their use of exact logical formulae for deriving results. The analysis of the bald man paradox above shows how an SUL, and hence probability and fuzzy logics, can adequately represent approximate or uncertain reasoning. From the underlying axiomatic structure of these logics it is also possible to show how the numeric truth values arise in a simple and natural way.

It is convenient to state this in terms of the resolution of a further paradox of classical logic, that of Russell's barber who shaves everyone in a village who do not shave themselves—does he shave himself? This corresponds to the fault in Frege's axiom of comprehension in naive set theory that Russell discovered—that not all predicates define sets (Kneale & Kneale, 1962, Chapter 11). It led to Russell's theory of types (Copi, 1971) as a solution to the problem that retained the structure of classical logic without leading to inconsistencies in set theory by placing restrictions on those predicates which could define sets. It has long been suggested that an alternative approach to the problem is to change the rules of classical logic, but this has proved difficult. Maydole (1975) showed that variants of Russell's paradox could be generated for virtually all variant logics except two attributed to Post and Lukasiewicz. These two cases were left open as Maydole's technique did not generate paradoxes but it was not possible to show using his methodology that some other technique might not do so. White (1979) showed that the quantified form of Lukasiewicz infinitely-valued logic did lead to a consistent set theory having a completely unrestricted axiom of comprehension—any predicate defines a set.

Analysing Russell's barber paradox illustrates the way in which new truth values are generated in an SUL. If we take S to mean that the barber shaves himself then the following two lines of reasoning apply: if S is true then the barber shaves himself so that he is not shaved by the barber and hence S is false, i.e.

$$\mathrm{S} \supset \bar{\mathrm{S}} \tag{19}$$

however, if S is false then the barber does not shave himself so that he is shaved by the barber and hence S is true, i.e.

$$\bar{\mathrm{S}} \supset \mathrm{S}. \tag{20}$$

These taken together give us that

$$\mathrm{S} \equiv \bar{\mathrm{S}}, \tag{21}$$

which means that S is an impossible proposition for classical logic—it is both true and false. However, having already noted that the law of contradiction does not hold in an SUL, we should not be surprised to find that S is a perfectly legitimate proposition in that logic. Indeed from relation (21) and (P14) we may derive that

$$p(\mathrm{S}) = p(\bar{\mathrm{S}}) = 1 - p(\mathrm{S}), \tag{20}$$

so that

$$p(\mathrm{S}) = 1/2. \tag{22}$$

Now this should be a surprising result because nowhere in the definition of an SUL have we defined the number 1/2. It has arisen from the rules given on the basis that if there is a proposition S with a truth value that satisfies the predicates defined by Russell and the rules of an SUL then its truth value must be 1/2. Thus the *assumption of existence* of such a proposition has generated the need for, and effectively *brought into being*, the intermediate truth value, 1/2. Varela (1975) has used this same line of reasoning to bring a third truth value into Brown's (1969) logic of distinctions in order to model the phenomena of self-reference.

The technique of using what are paradoxes of classical logic to generate truth values in a multivalued logic can be extended to give a generator for any required value. Consider an expression of the form of equation (20), $\bar{x} \supset x$, then from (P13) and (P14)

$$p(\bar{x} \supset x) = 1 - p(\bar{x}) + p(x \wedge \bar{x}) = p(x) + p(x \wedge \bar{x}). \tag{24}$$

However, from relation (1)

$$p(x \wedge \bar{x}) \leq \min\,(p(x), p(\bar{x})) = \min\,(p(x), 1 - p(x)), \tag{25}$$

so that

$$p(x) \leq 1/2 \rightarrow p(x \wedge \bar{x}) = p(x) \tag{26}$$

and hence

$$p(x) \leq 1/2 \rightarrow p(x) = p(\bar{x} \supset x)/2. \tag{27}$$

Now consider a series of propositions, $\mathrm{S}(i)$, that are defined recursively by

$$(\bar{\mathrm{S}}(i) \supset \mathrm{S}(i)) \equiv \mathrm{S}(i-1), \tag{28}$$

with

$$S(0) \equiv F \supset F, \quad \text{and} \quad S(1) \equiv S, \tag{29}$$

where S is the proposition defined by relation (21). Then we have from relation (27) that

$$p(B(i)) = (1/2)^i, \tag{30}$$

so that the assumption of the existence of the S(i) generates an infinite sequence of propositional constances with truth values having binary fractional powers, in place value notation of the form 0·00001000000 . . . in radix 2 for some position of one in a sequence of zeros.

This may be extended to the generation of an arbitary truth value by considering an expression of the form

$$R = (\bar{S}(i) \supset (\bar{S}(j) \supset (\bar{S}(k) \supset (\cdots \supset S(n))))), \tag{31}$$

where $i < j < k < \cdots < n$. Then

$$p(R) = (1/2)^i + (1/2)^j + (1/2)^k + \cdots + (1/2)^n. \tag{32}$$

Hence we may use the propositional sequence S(i) to generate any truth value by expressing it as a fractional binary expansion and putting the proposition S(i) in an expression of the form of equation (31) if there is a 1 in the ith binary place of the expansion. That this derivation applies to an SUL in general, and hence to both fuzzy and probabilistic logics, again emphasizes the link between them since the modelling of the modal logic S5 through infinite binary sequences is a well-known result in temporal logic (Rescher, 1969, p. 194).

This derivation of numeric values from the basic axioms of the system, which themselves introduce only the numbers 0 and 1, is a significant indicator of the source of numeric truth values in uncertain reasoning. They may be seen as arising out of an *order* relation on propositions, which may be one of *probability*, *credibility*, *plausibility* or some such term dependent on how we view the situation creating uncertainty. Under uncertainty we may not be able to make clear-cut assignments of truth or falsity to statements. However, logical constraints may lead us to assertions of the form that "even though neither x nor y is definitely true or false, x is more probable (credible, plausible) than y". This relation can be expressed by assigning x some non-zero truth value and y some greater, non-unity truth value. Now if a further statement is made that is more probable (credible, plausible) than x and less than y then it needs to be placed between them. Thus, in terms of our fractional binary expansions, longer and longer sequences may become necessary to cope with the distinctions to be made in ordering the statements.

The use of numbers implies that we regard the uncertainties about statements as being well-ordered. However, the arguments of this section apply also to the partially ordered algebraic structures that Goguen (1981) introduces as generalized truth-sets. We need some mechanism of interpolation if we are to encode the differentiations we wish to make in our statements about the world and that is how the numbers arise from the logical structure of our statements.

Arithmetic mean in a SUL

As a final example of the way in which some of the stereotyped distinctions between fuzzy and probabilistic logics have no basic foundation but arise out of use, consider the arithemetic mean, or averaging operator over a set of truth values. This arises naturally in statistical operations involving probabilities but seems unnatural in fuzzy logic where a min or max operation is a more "natural" way of combining numbers. However, the results of relations (24)–(27) can be extended to give a natural averaging operating in a SUL and hence in both fuzzy and probability logics. Suppose we have n propositions $X(i)$, $1 \leq i \leq n$ then we can first use an extended version of equation (24), analogous to equation (31), to implicitly define related propositions $Y(i)$:

$$X(i) = (\bar{Y}(i) \supset (\bar{Y}(i) \supset (\bar{Y}(\cdots)))), \tag{33}$$

where there are $n\,Y(i)$ on the right-hand side, so that

$$p(Y(i)) = p(X(i))/n. \tag{34}$$

Then we use the $Y(i)$ in an expression again analogous to equation (31) to define a proposition Z:

$$Z = (\bar{Y}(1) \supset (\bar{Y}(2) \supset (\bar{Y}(3) \supset (\cdots \supset Y)n)))), \tag{35}$$

so that

$$\begin{aligned} p(Z) &= p(Y(1)) + p(Y(2)) + p(Y(3)) + \cdots + p(Y(n)) \\ &= (p(X(1)) + p(X(2)) + p(X(3)) + \cdots + p(X(n)))/n. \end{aligned} \tag{36}$$

Thus the arithmetic mean of a set of truth values may be constructed within a SUL and hence is a legitimate operation of both fuzzy and probability logics.

Sequential system identification

As a final example that shows up the different roles of the fuzzy and probabilistic derivations from a SUL consider the problem of sequential system identification where the data is uncertain. This is a classical system-theoretic problem where the behaviour of a system is given as a time sequence and one attempts to derive the structure of the system producing it (Gaines, 1978*b*). Thus one might have a sequence of symbols representing observations such as XYXYYYYYXY and wish to derive a deterministic, non-deterministic or stochastic automata that could have generated that sequence.

The identification problem can be formulated as: given a class of models, M, and a class of behaviours, B, derive a relation between B and M that ascribes one, or more, models in M to a given behaviour in B such that the relation is optimal under certain additional given constraints. These additional constraints are concerned with measures of approximation between models and behaviours and preference orderings of complexity on models. This problem was solved for deterministic automata by Nerode (1958) and given a category theoretic formulation for general deterministic systems by Goguen (1973), Arbib & Manes (1974) and Ehrig (1974). It was solved by Gaines (1975*b*, 1977) for stochastic automata and given a category theoretic formulation by Ralescu (1979) for general non-deterministic systems.

A particularly interesting identification problem for this paper is that where the observation sequence is itself uncertain. Suppose the behaviour to be modelled is XYXYZZZYXY where the Z's represent uncertainty about whether an X or a Y was observed. There is no mutual exclusion involved in the observation statements and we might represent a specific situation by a series of degrees of membership:

	1	2	3	4	5	6	7	8	9	10
X	1·0	0·0	1·0	0·0	0·6	0·8	0·7	0·0	1·0	0·0
Y	0·0	1·0	0·0	1·0	1·0	1·0	1·0	1·0	0·0	1·0

Thus, the sixth observation statement says that Y possibly was observed with degree of membership 1·0 and X possibly was observed with degree of membership 0·8. If we had our eyes closed at the time both would be 1·0, but the statement implies that we have less evidence for X than for Y.

Now, this uncertainty about the behaviour also induces an uncertainty about the model for it. There are various possible models for this sequence and they are not mutually exclusive. However, if we take the models themselves to be automata then the states of the model are mutually exclusive. This is a result of the semantics which we impose on the notion of state, that a system can be in only one state at a time. Gaines & Kohout (1975) show that the conventional model of a non-deterministic automata is inadequate to cope with these semantics and that a probabilistic model must be used that corresponds to the modal logic S5—the required constraint is that the automaton must be in one, and one only, state at a time. This gives us a mixed result in that we have a (fuzzy) possibilistic distribution over models which themselves have a (probability) possibilistic distribution over states.

Gaines (1979) gives a solution to modelling the above sequence as three stochastic grammars each with a differing degree of membership:

M1: membership = 0·6, approximation = 0·300,
$a \to Xb \quad (p = 1{\cdot}0) \qquad b \to Ya \quad (p = 1{\cdot}0)$;
M2: membership = 0·7, approximation = 0·693,
$a \to Xb \quad (p = 0{\cdot}8) \qquad a \to Yb \quad (p = 0{\cdot}2) \qquad b \to Ya \ (p = 1{\cdot}0)$;
M3: membership = 1·0, approximation = 0·817,
$a \to Xb \quad (p = 0{\cdot}6) \qquad a \to Yb \quad (p = 0{\cdot}4) \qquad b \to Ya \quad (p = 1{\cdot}0)$.

M3 may be seen as a model based on taking the most certain observations and then deriving the best fit to them. It leaves a high degree of probabilistic uncertainty in the model. M1 is at the other extreme and removes this uncertainty altogether by precisifying the data, saying that observation 5 should be taken as an X rather than a Y. Note that all three models are valid in some sense—we cannot say more about possible models without making further assumptions which are not inherent in the data as presented.

The essential mixture of fuzzy and probabilistic uncertainty involved shows up well if we consider the predictions of each model at each step. All three models agree that observation 1 was X and 2, 4, 6, 8 and 10 were Y—they agree on resolving the uncertainty about observation 6. However, M2 and M3 introduce uncertainty in the models about what could have happened at 3 and 9 when there was no uncertainty about the data. If we combine all three models then the prediction as to the observation expected at 2, 4, 6, 8, 10 is Y with degree of membership 1·0 and probability 1·0. However, the prediction at 1, 3, 5, 7, 9 is more complex and may be represented as

$$X((1{\cdot}0, 0{\cdot}6), (0{\cdot}8, 0{\cdot}7), (0{\cdot}6, 1{\cdot}0)),$$
$$Y((0{\cdot}0, 0{\cdot}6), (0{\cdot}2, 0{\cdot}7), (0{\cdot}4, 1{\cdot}0)),$$

where the interior pairs are a probability together with its degree of membership. Hence the predictions of X and Y have become *possibility vectors* to use Zadeh's (1978) terminology.

This example shows the differing roles that the fuzzy and probabilistic resolutions of a SUL possibility logic have in modelling differing forms of possibilistic uncertainty. It also demonstrates that the different logics are not competitive or mutually exclusive and that combinations are necessary to represent even a fairly simple modelling situation.

Conclusions

One major conclusion to be drawn from this paper should now be obvious—that fuzzy sets theory and probability theory should be viewed not as rivals, and not necessarily even as complementary, but rather as similar logical systems, having a common core that is adequate for many aspects of systems analysis and design, and differing in certain well-defined features that may, or may not, be relevant in particular applications.

A standard uncertainty logic (SUL) based on a system of valuations on a lattice subsumes both probability and fuzzy logics and enables key inference rules common to both to be derived. The logics only diverge in a final step whereby the use of fuzzy logic assumes strong functionality and the use of probability logic assumes the law of the excluded middle. This is not to suggest that the difference between the two logics is trivial, but rather to show that it does not necessarily lie in major part in the connectives used. For many applications an SUL is adequate and this encompasses both forms of connectives.

In terms of generating general systems methodologies (Gaines & Shaw, 1981) that can commence with primitive notions of making distinctions and build up the whole structure of ontology, epistemology and axiology required for both mathematical theory and engineering practice the notion of a SUL presented here seems to take us one further step towards a unified theory of systems. I hope the presentation has been such that the key results common to both fuzzy and probability logics stand out as obvious both in derivation and implication.

Returning to my initial theme it seems clear that the past is fuzzy also, not only in the history of fuzzy sets theory over the past twenty years, but also technically when we set up modelling schemata as in the previous section. The precision of the past, and the precision required in past methodologies, can both be seen to be myths, very significant myths that have generated much that is fundamentally important. However, there comes a time when any myth has served its purpose and we must put away dogma and look at reality with fresh eyes if we are to see into greater distances and greater depths. We now have the tools to control the effects of the precisiation and to do away with an artificially precise past, if that is what is necessary to lead us into a fuzzy, but realistic, future.

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