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Fuzzy and Probability Uncertainty Logics

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Probability theory and fuzzy logic have been presented as quite distinct theoretical foundations for reasoning and decision making in situations of uncertainty. This paper establishes a common basis for both forms of logic of uncertainty in which a basic uncertainty logic is defined in terms of a valuation on a lattice of propositions. The (non-truth-functional) connectives for conjunction, disjunction, equivalence, implication, and negation are defined in terms which closely resemble those of probability theory. Addition of the axiom of the excluded middle to the basic logic gives a standard probability logic. Alternatively, addition of a requirement for strong truth-functionality (truth-value of connective determined by truth-value of constituents) gives a fuzzy logic with connectives, including implication, as in Lukasiewicz' infinitely valued logic. A common semantics for all such variants is given in terms of binary responses from a population. The type of population, e.g., physical events, people, or neurons, determines whether the model is of physical probability, subjective belief, or human decision-making. The formal theory and the semantics together illustrate clearly the precise similarities and differences between fuzzy and probability logics.

1. INTRODUCTION

Multivalued logics using truth-valuations in the interval [0, 1] and min/max connectives for conjunction/disjunction have been presented in the literature in recent years as nonprobabilistic, "fuzzy" logics which are favored alternatives to probability theory in explicating some aspects of imprecise and uncertain concepts and decisions (Bellman and Zadeh, 1970, p. 141; Coguen, 1969, p. 340; Lee, 1972, p. 109; Sanford, 1975, p. 31). In classical studies (Rescher, 1969) these multivalued logics have had such different domains of application from probability logic that there has been little incentive to make detailed comparisons between the two approaches. However, particularly in view of recent critical comment about the role of fuzzy logics where probability theory might be applied (Arbid, 1977; Fox, 1977; Stallings, 1977) and practical comparisons between them (Gaines, 1975; Baas and Kwakernaak, 1977) it now seems essential to establish the exact relationships, similarities and differences, between probability and fuzzy logics.

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This paper develops a basic uncertainty logic in terms of a valuation on a lattice of propositions that is a common foundation to both fuzzy and probability logics. Two additional, and incompatible, axioms are then proposed, one of which leads to Lukasiewicz' infinitely valued "fuzzy" logic, while the other leads to classical probability logic. A semantic model is also given of the basic uncertainty logic which can be interpreted in a variety of ways to illustrate the similarities and differences between fuzzy and probability logics.

Two specific points are worth emphasizing before the technical presentation. First, that this paper is not reductionist—there *are* significant differences between fuzzy logics and probability logics, in their motivations, applications, and axioms. However, there are also close relationships between the two forms of logic which are themselves significant. Second, the term *fuzzy logic* has been used variously in the literature to denote:

(a) *A basis for reasoning with vague statements.* The term "fuzzy" had a colloquial meaning before Zadeh gave it a technical definition and there are (Gaines and Kohout, 1977) *independent uses* of the term; *deliberately variant* fuzzy logics; and *unwitting variations* (Arbib, 1977);

(b) *A basis for linguistic reasoning with vague statements using fuzzy set theory for the fuzzification of logical structures.* This more restricted definition corresponds to Zadeh's own papers where the emphasis is on *linguistic truth values* (Bellman and Zadeh, 1977);

(c) *A multivalued logic in which truth values are in the interval* [0, 1], *the valuation of a disjunction is the maximum of the disjuncts, and that of a conjunction is the minimum of the conjuncts.* This is a "population stereotype" for fuzzy logics. It may be specialized to the form of implication being that of Lukasiewicz, infinitely valued logic (Giles, 1975), or generalized to truth-values in a lattice (Goguen, 1969; Brown, 1971). All variants of this definition have in common an ordered set of truth-values and define the logical connectives in terms of this ordering.

There is ample scope for confusion since, for example, Zadeh (1975) fuzzifies in terms of (b) logics that are already fuzzy in the sense of (c). In this paper I am concerned primarily with definition (c), bearing in mind however that the key requirement is the use of fuzzy logics as a basis for linguistic reasoning as in (b).

2. MIN/MAX CONNECTIVES IN PROBABILITY LOGICS

This section shows that the min/max connectives that are often supposed to characterize fuzzy logics also arise in probability logic. It is intended to give an intuitive feeling for the relationship between the logics in preparation for the more detailed analysis of Section 3. For the moment a fuzzy logic is taken to have operations of conjunction, disjunction, and negation as defined by Lee (1972) and the valuation of implication is not considered. In these terms fuzzy logic may be thought of as a multivalued extension of Boolean logic based on Zadeh's (1965) fuzzy set theory in which truth-values are extended from the end points of the interval, [0, 1], to range through the entire interval. The normal logical operations are defined in terms of arithmetic operations on these values regarded as "degrees of membership" to truth. That is (taking a lowercase letter as a logic variable, and the corresponding capital letter as its degree of membership):

$$z = x \text{ AND } y \Rightarrow Z = \min(X, Y), \tag{1}$$

$$z = x \text{ OR } y \Rightarrow Z = \max(X, Y), \tag{2}$$

$$z = \text{NOT } y \Rightarrow Z = 1 - Y. \tag{3}$$

These definitions coincide with the normal logic functions for the two extreme values (TRUE = 1, FALSE = 0).

Using the same notation as above but regarding, for example, X as being not only a degree of membership but also the actual probability of occurrence of event x, one may derive the probabilistic equivalents of Eqs. (1) through (3). It is assumed that the events themselves are binary in nature and either occur or do not occur. Equation (3) still applies (as usual, $\bar{x}$ means the nonoccurrence of x).

For

$$z = \text{NOT } y, \qquad Z = p(z) = p(\bar{y}) = 1 - p(y) = 1 - Y. \tag{4}$$

Consider now the expressions for X and Y in terms of the joint probabilities of events x and y:

$$X = p(x) = p(x \wedge y) + p(x \wedge \bar{y}), \tag{5}$$

$$Y = p(y) = p(x \wedge y) + p(\bar{x} \wedge y). \tag{6}$$

From these two equations, given that probabilities lie in the interval [0, 1], we may derive the inequalities

$$0 \leqslant p(x \wedge y) \leqslant \min(X, Y), \tag{7}$$

$$0 \leqslant XY \leqslant \min(X, Y), \tag{8}$$

$$\max(X, Y) \leqslant p(x \vee y) \leqslant 1, \tag{9}$$

$$\max(X, Y) \leqslant X + Y - XY \leqslant 1. \tag{10}$$

Consider now the significance of each of the three values in these inequalities being attained:

For $z = x$ AND y, $Z = p(x \wedge y)$, the conditions are:

(i) $Z = 0 \Leftrightarrow p(x \wedge y) = 0 \Leftrightarrow x \supset \bar{y}$ AND $y \supset \bar{x}$, i.e., x and y are mutually exclusive.

(ii) $Z = XY \Leftrightarrow p(x \wedge y) = p(x)\,p(y)$, i.e., x and y are statistically independent.

(iii) $Z = \min(X, Y) \Leftrightarrow p(x \wedge \bar{y}) = 0$ OR $p(\bar{x} \wedge y) = 0 \Leftrightarrow x \rightarrow y$ OR $y \rightarrow x$, i.e., one of x and y strictly implies the other.

For $z = x$ OR y, $Z = p(x \vee y) = 1 - p(\bar{x} \wedge \bar{y})$, the conditions are:

(i′) $Z = 1 \Leftrightarrow p(\bar{x} \wedge \bar{y}) = 0 \Leftrightarrow \bar{x} \supset y$ AND $y \supset \bar{x}$, i.e., one of x and y must occur.

(ii′) $Z = X + Y - XY \Leftrightarrow p(x \wedge y) = p(x)\,p(y)$, i.e., x and y are statistically independent.

(iii′) $Z = \max(X, Y) \Leftrightarrow p(x \wedge \bar{y}) = 0$ OR $p(\bar{x} \wedge y) = 0 \Leftrightarrow x \rightarrow y$ OR $y \rightarrow x$; i.e., one of x or y strictly implies the other. Note the emphasis on *strict*, rather than material, implication: $x \rightarrow y$ is equivalent to *necessarily* $x \supset y$. The weaker result, $p((x \supset y) \vee (y \supset x)) = 1$, is always true.

It can be seen that conditions (i) and (i′) are independent, and together imply that $x = y$. Conditions (ii) and (ii′) are equivalent and together lead to a probability logic in which atomic propositions are assumed statistically independent, giving multiplication and addition as connectives. Conditions (iii) and (iii′) are equivalent and together lead to a probability logic with the max/min connectives of a "fuzzy" logic between atomic propositions which are assumed to form a single chain of implication. Thus, informally, the assumptions leading to these very different forms of logical connective are seen to be of opposite nature (independence versus implication), but both multiplication/addition, and max/min, connectives may be seen to arise from constraints on an underlying probability logic.

3. A Formal Basis for the Comparison of Various Logics of Uncertainty

The main function of the comparison derived in Section 2 was to demonstrate that (as noted by Gaines, 1975; Gaines and Kohout ,1975; Watanabe, 1975) the use of min and max operations in fuzzy logic is not sufficient to discriminate the logic from that of probability theory—both these operations arise naturally in the calculation of the conjunction and disjunction of probabilistic events. Our association of addition and multiplication as natural operations upon probabilities comes from our frequent interest in statistically independent events, not from the logic of probability itself. However, the derivation so far has some anomalies which make it necessary to probe rather deeper to discover the exact nature of the relationship between fuzzy and probability logics. In particular, Eq. (5) enables one to derive (by substituting x for y) the result that $p(x \wedge \bar{x}) = 0$, the law of contradiction, a thesis which is not one of fuzzy logic as normally defined. The difficulty arises because in terms of the lattice of statements about events, negation in probability logic is a true complement, whereas in fuzzy logic it is not even a pseudocomplement (Birkhoff 1948, p. 147).

In this section the analysis of Section 2 will be repeated more formally, and a logic will be developed based on a valuation over a lattice that closely resembles probability logic (I shall call it an *uncertainty logic* (UL)), but in which neither the law of the excluded middle (LEM) nor that of contradiction are tautologies. This logic will be shown to default to the standard propositional calculus (PC) when truth-values are restricted to be 0 or 1, and hence to be true extension of PC. A definition of logical equivalence in terms of a metric on the lattice will be used to define implication and negation. Given this logic, it will be shown that the addition of either one or the other of what, in Heyting's intuitionistic propositional calculus (IPC) terms are "paradoxes" of PC, (Rescher, 1968, Chap. 2) leads to either a probability or a fuzzy logic. The assumption of $p(x \vee \bar{x}) = 1$ gives precisely Rescher's (1968, Chap. 11) probability logic, while the assumption that either $p(x \supset y) = 1$ or $p(y \supset x) = 1$ gives a fuzzy logic which is precisely Lukasiewicz $L_{\aleph}$. (Rescher 1968, Chap. 6).

Let $L(X, F, T, \vee, \wedge)$ be the free lattice generated by a set of elements, X, under two (idempotent, commutative) semigroup operations, $\vee$, $\wedge$, with maximal element T and minimal element F, i.e., L satisfies

(P1) $\forall x \in L, \quad x \vee x = x \wedge x = x,$

(P2) $\forall x, y \in L, \quad x \vee y = y \vee x, \; x \wedge y = y \wedge x,$

(P3) $\forall x, y, z \in L, \quad x \vee (y \vee z) = (x \vee y) \vee z, \; x \wedge (y \wedge z) = (x \wedge y) \wedge z,$

(P4) $\forall x, y \in L, \quad x \vee (x \wedge y) = x, \; x \wedge (x \vee y) = x,$

(P5) $\forall x \in L, \quad x \vee T = T, \; x \wedge T = x, \; x \vee F = x, \; x \wedge F = F,$

the idempotent, commutative, associative, and adsorption postulates, together with a definition of the minimal and maximal elements (Birkhoff, 1948, p. 18). The usual order relation may also be defined:

(P6) $\forall x, y \in L, \quad x \leqslant y \Leftrightarrow \exists z \in L : y = x \vee z.$

It is possible to make cases for weaker structures (e.g., dropping idempotency) but, for present purposes, this will be taken as an unreasonably wide generalization of our concepts of conjunction and disjunction. Now suppose that every element of L is assigned a "truth-value" (for different applications, different terminologies may be more appropriate, "probability", "degree of knowledge," "level of belief," etc.) in the closed interval [0, 1], by a continuous, order-preserving function, $p: L \rightarrow [0, 1]$, with the constraints

(P7) $p(F) = 0, \quad p(T) = 1;$

(P8) $\forall x, y \in L, \quad x \leqslant y \Rightarrow p(x) \leqslant p(y);$

(P9) $\forall x, y \in L, \quad p(x \wedge y) + p(x \vee y) = p(x) + p(y),$

i.e., p is a continuous, order-preserving *valuation* (Birkhoff, 1948, p. 74) on L. Note that for p to exist the lattice must be modular, and that we have

$$p(x \wedge y) \leqslant \min(p(x), p(y)) \leqslant \max(p(x), p(y)) \leqslant p(x \vee y). \qquad (11)$$

Now the relation defined by

(P10) $\forall x, y \in L, \quad x \equiv y \Leftrightarrow p(x \wedge y) = p(x \vee y)$

is a *congruence* on L (Birkhoff, 1948, p. 77), so that

$$\forall x, y, z \in L, \quad x \equiv y \Leftrightarrow (x \wedge z) \equiv (y \wedge z) \text{ AND } (x \vee z) \equiv (y \vee z), \quad (12)$$

which in its turn implies

$$\forall x, y, z \in L, \quad p(x \wedge y) = p(x \vee y) \Rightarrow p(x \wedge z) = p(y \wedge z) \text{ AND } p(x \vee z) = p(y \vee z), \quad (13)$$

i.e., $x \equiv y$ means that y may be substituted freely for x in p expressions without changing their value. Thus, with respect to the valuation p, the relation, $\equiv$, is one of *logical equivalence.*

To give the final touch to this structure as a multivalued logic we need to define implication and negation. It is worth pondering these because the general structure so far is common to virtually all the logics, for example, in Rescher (1969) it is largely the definitions of implication and negation that generates a particular multivalued logic. Note, for example, that P8 and P9 together are adequate to ensure that there are unambiguously defined truth tables for conjunction ($\wedge$) and disjunction ($\vee$) in the binary case when the domain of p is restricted to the end points of the interval, and that these are identical to those of the normal propositional calculus. These two postulates also enable us to infer the inequalities of Eq. (11), an intuitively satisfying result. Clearly the postulate, P9, still holds when the outer inequalities become equalities—a further demonstration that the additivity of probability-like valuations is completely compatible with, and indeed closely related to, the min/max connectives of fuzzy logic.

Equivalence, Implication, and Negation in Metric Terms

The most elegant and intuitively satisfying route to definitions of implication and negation in the present context is through the definition of a metric on the lattice giving a measure of the "distance" apart of two propositions under a valuation. This is naturally based on the congruence already defined since it is clearly desirable that congruent elements, being logically equivalent under the valuation, should be at zero distance from one another. Define:

(P11) $\forall x, y \in L, \qquad d(x, y) = p(x \vee y) - p(x \wedge y).$

Birkhoff (1948, p. 77) shows that d defines a quasimetric on L such that

$$d(x, x) = 0, \tag{14}$$

$$0 \leqslant d(x, y) \leqslant 1, \tag{15}$$

$$d(x, y) + d(y, z) \geqslant d(x, z). \tag{16}$$

Additionally, P10 ensures that d defines a true metric on the quotient lattice under the congruence already defined so that

$$d(x, y) = 0 \Leftrightarrow x \equiv y. \tag{17}$$

This is not the only metric on a lattice upon which logics may be based but it is the one that generates both probability and fuzzy logics so that it suffices for this paper.

The distance defined by d varies from 0 to 1 with 0 implying that two elements are congruent. For consistency with the interpretation of the valuation itself, where 1 means true, it is convenient to define a measure of equivalence, $p\ (x \equiv y)$, between two members of L as 1 minus the distance between them:

(P12) $\forall x, y \in L \qquad p(x \equiv y) = 1 - d(x, y)$
$\qquad\qquad = 1 - p(x \vee y) + p(x \wedge y).$

Thus two congruent elements are equivalent with a valuation of 1, while two maximally nonequivalent elements (congruent to T and F) are "equivalent" with valuation 0. This consistency between the valuation of equivalence and that of lattice elements becomes important if we should wish to postulate lattice elements (rather than a purely metalinguistic measure) representing the equivalence, implication, and negation of lattice elements.

We may now define a valuation of the extent to which x "implies y" by noting that if $x \supset y$ is true in the PC sense in L, then $x \wedge y = x$ and $x \vee y = y$. The degree of equivalence between $x \wedge y$ and x (or $x \vee y$ and y—they turn out to be the same) is a suitable measure of the strength of implication:

(P13) $\forall x, y \in L, \qquad p(x \supset y) = p(x \equiv x \wedge y) = 1 - d(x, x \wedge y)$
$\qquad\qquad = 1 - p(x) + p(x \wedge y)$
$\qquad\qquad = 1 + p(y) - p(x \vee y) = 1 - d(y, x \vee y).$

We may now go on to define negation in the usual way (Prior, 1962, p. 50) in terms of equivalence or implication:

(P14) $\forall x \in L, \qquad p(\bar{x}) = p(x \equiv F) = p(x \supset F) = 1 - p(x).$

One may note that conversely,

$$p(T \equiv x) = p(T \supset x) = p(x), \tag{18}$$

as it should.

The form of implication defined in P13 has the property that

$$\forall x, y \in L, \qquad p(y) = p(x \vee y) - 1 + p(x \supset y) \geqslant p(x) - (1 - p(x \supset y)), \tag{19}$$

which enables a lower bound to be placed on the truth value of y given those for x and $x \supset y$, thus allowing a limited form of *modus ponens*. Thus it satisfies the normal requirement (Lee and Chang, 1971) that the assertion of $x \supset y$ may be used to infer that $p(y) \geqslant p(x)$, and hence also that $p(y) \geqslant \max(p(x_i))$ where y is constrained by "rules" of the form $x_i \supset y$ ("if x_i then y"), a common pattern of inference in, for example, control applications of fuzzy reasoning (Mamdani and Assilian, 1975).

These definitions again lead to the truth tables of PC in the binary case, and it is perhaps worth noting that they are a counterexample to Lee's (1972) remark that, "by rejecting the evaluation procedure of fuzzy logic, one would simultaneously reject that of two-valued logic"—long before the logic has been specialized to a particular set of truth-functional connectives it reduces to PC in the binary case.

I have deliberately left open the question of whether there are elements within L that represent $x \supset y$ and $\bar{x}$—much of what one wishes to use in either probability or fuzzy logics does not require this assumption, e.g., models of state-determined machines, automata, involve only the connectives for conjunction and disjunction, not those for implication and negation (Gaines and Kohout, 1975). However, let us assume for the moment that there are elements in L representing $x \equiv y$, $x \supset y$, and $\bar{x}$. Then we have from (P10 and P13):

$$p(x \vee (x \supset y)) + p(x \wedge (x \supset y)) = 1 + p(x \wedge y), \tag{20}$$

so that

$$p(x \vee \bar{x}) + p(x \wedge \bar{x}) = 1. \tag{21}$$

Thus the law of the excluded middle ($p(x \vee \bar{x}) = 1$) and the law of contradiction ($p(x \wedge \bar{x}) = 0$) are equivalent in this logic in that postulating one implies the other.

The structure developed so far is the common denominator of probabilistic and fuzzy logics—each will be derived from it in the following sections. In summary, P1 through P5 define a lattice, and P6 through P9 a positive isotone valuation upon it. I have previously (Gaines, 1976) called this a "basic probability logic" because it satisfies all the normal postulates of a probability algebra (Birkhoff, 1948, p. 197) except that negation and its valuation are undefined. However, this terminology is potentially confusing given that probability valuations are usually defined only on complemented distributive lattices, and the term used in this paper, a *basic uncertainty logic* (for a system satisfying P1 through P11) seems better. The valuations of equivalence, implication, and negation, defined by P12 through P14, are not the only ones possible (see Gaines (1976) for alternatives related to conditional probability), but are very natural so that P1 through P14 might fairly be said to define a *standard uncertainty logic* (SUL).

Note that a standard uncertainty logic leads to the sum of the values of an element and its negation being unity (P14), which is the remaining postulate for a probability on a language adopted, for example, by Fenstad (1967). Up to the point of choosing this form of negation, our basic uncertainty logic could have been specialized into Heyting's intuitionistic propositional calculus by postulating pseudocomplementation in L. However, for example, Eq. (21) is inconsistent with IPC. Note that the only implied constraint on the lattice is that the quotient lattice under the congruence be modular (Birkhoff, 1948, p. 76)—it need not be distributive, complemented, or pseudocomplemented. We can add a postulate of distributivity:

(P15) $\forall x, y, z \in L, \qquad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

which implies the other forms of distributivity (Birkhoff, 1948, p. 133) and is needed for the two specializations of the SUL to be described in the following sections.

4. Derivation of Rescher's Probability Logic

Rescher (1969, p. 185) defines a probability logic over a domain of statements in the propositional calculus in terms of a function, p, that assigns to a statement, x, a real value, $p(x)$ satisfying the following postulates:

(R1) $0 \leqslant p(x)$, for any statement x;

(R2) $p(x \vee y) = p(x) + p(y)$, provided that x and y are mutually exclusive;

(R3) $p(x) = p(y)$ when x and y are logically equivalent;

(R4) $p(x \vee \bar{x}) = 1$.

From these one may derive the further results:

$$0 \leqslant p(x) \leqslant 1, \tag{22}$$

$$p(F) = 0, \qquad p(T) = 1, \tag{23}$$

$$p(x \wedge y) \leqslant \min(p(x), p(y)) \leqslant \max(p(x), p(y)) \leqslant p(x \vee y), \tag{24}$$

$$p(\bar{x}) = 1 - p(x), \tag{25}$$

$$p(x \wedge y) = p(x) + p(y) - p(x \vee y), \tag{26}$$

$$\begin{aligned} p(x \supset y) &= p(\bar{x} \vee y) \\ &= p(\bar{x}) + p(x \wedge y), \end{aligned} \tag{27}$$

$$\begin{aligned} p(x \equiv y) &= p((x \supset y) \wedge (y \supset x)) \\ &= p((\bar{x} \vee y) \wedge (\bar{y} \vee x)) = 1 - p(x \vee y) + p(x \wedge y), \end{aligned} \tag{28}$$

where T is any statement asserted and F is any statement whose negation is asserted. This is the same system as that taken by Fenstad (1967) to represent probabilities on a first order language, but he takes (R3) and Eqs. (23)–(26) as postulates.

If we take the domain of statements to be defined as a lattice $L(X, T, F, \vee, \wedge)$ satisfying P1 through P4, then R1, R2, and R3 are clearly related to P6 through P9. R4 is shown by (21) not to be a tautology, but to be equivalent to the assumption of either the law of the excluded middle or that of contradiction. We now prove that adding either law to a distributive standard uncertainty logic as previously defined results in a system equivalent to Rescher's probability logic.

THEOREM 1. *The addition of the following postulate,* P16, *to those for a distributive standard uncertainty logic,* P1 *through* P15, *gives a logic identical to Rescher's probability logic* (*as defined by* R1 *through* R4).

(P16) $\forall x \in L, \qquad p(x \vee \bar{x}) = 1.$

Proof. First consider whether Rescher's postulates lead to P1 through P15: P1 through P6 and P15 follow since the language over which a probability is defined is PC; P7 is (23); P8 follows from (24); P9 is (26); P10 follows from (28); P11 is a definition; P12 is (28); P13 is (27); P14 is (25); and P16 is R4. Conversely, R1 through R4 follow from P1 through P16.

Thus adding LEM to a distributive SUL gives a conventional probability logic. In Section 5 it is shown that an alternative addition gives a fuzzy logic.

5. DERIVATION OF $Ł_{\aleph_1}$, A FUZZY LOGIC

The multivalued logic which Zadeh (1975) takes as a basis for his model of linguistic reasoning with vague statements is Lukasiewicz' infinitely valued logic, $Ł_{\aleph_1}$ (Rescher, 1969, Sect. 6), the connectives of which are defined entirely in terms of truth rules. For any statement, x, we have a truth-value, $p(x)$, such that

(L1) $0 \leqslant p(x) \leqslant 1$.

The logical connectives are then defined by

(L2) $p(\bar{x}) = 1 - p(x)$,

(L3) $p(x \wedge y) = \min(p(x), p(y))$,

(L4) $p(x \vee y) = \max(p(x), p(y))$,

(L5) $p(x \supset y) = \min(1, 1 - p(x) + p(y))$,

(L6) $p(x \equiv y) = \min(1 - p(x) + p(y), 1 + p(x) - p(y))$.

Note that the connectives themselves are assumed to have no properties other than being linguistic markers. However, the quotient language under the equivalence of L6 is clearly a distributive lattice, but the complementation of L2 is nonstandard in that it does *not* define a complement in this lattice.

We can now prove that the standard uncertainty logic of P1 through P14 becomes $L_{\aleph_1}$ with the addition of an alternative postulate to P16, one of necessary implication between two propositions, P17:

THEOREM 2. *The addition of the following postulate,* P17, *to those for a distributive standard uncertainty logic,* P1 *through* P15, *gives a logic identical to Lukasiewicz' $L_{\aleph_1}$ as in* L1 *through* L6.

(P17) $\forall x, y \in L, \quad p(x \supset y) = 1$ OR $p(y \supset x) = 1$.

Proof. First if we assume L1 through L6, then P1 through P15 and P17 follow trivially. Conversely, P17 together with P13 and P8 allow us to infer L3 and the rest follow.

Note again that P17 is in terms of *necessary*, or strict, implication. We have in both probability and fuzzy logics that

$$p((x \supset y) \vee (y \supset x)) = 1. \tag{29}$$

Truth-Functionality

There is an alternative derivation of $L_{\aleph_1}$ from a SUL which throws further light on the result. Suppose we wish to make our SUL *strongly truth-functional* in the sense that for *any* elements $x, y \in L$, $p(x \vee y)$, $p(x \wedge y)$, etc., for all the connectives, are equationally defined in terms of $p(x)$ and $p(y)$. Then the arguments of Bellman and Giertz (1973) may be used to show that if the connectives are continuous in their arguments and P1 through P14 hold, so do L1 through L6. LEM is clearly inconsistent with such truth-functionality and if LEM is considered essential (Sanford 1975), then a weaker requirement for truth-functionality is necessary, typically that the values of connectives between lattice elements that have no generating element is common are equationally defined. The assumption of statistical independence in Section 2 gives such a weakly truth-functional SUL with LEM, but one where the value of a compound statement depends on its structure not just on the values of its components.

Whether there *are* situations in which strong truth-functionality can be reasonably demanded is clearly a semantic question. However, the simplicity of the resultant logic, if it can be assumed, is clearly attractive and is part of the explanation of the widespread attraction of "fuzzy logic."

6. Semantics for the Logics in Terms of Population Responses

The preceding sections have established a formal relationship between probability and fuzzy logics, and have demonstrated that their communality in terms of basic axioms and concepts is more substantial than their differences. The basic difference may be regarded as stemming from the postulation of LEM in one and strong truth-functionality in the other. A further distinction arises if a probability logic is specialized to a "stochastic logic" by making it truth-functional through the assumption of statistical independence. However, the actual significance of these similarities and differences can only be determined in terms of their semantics. Giles (1975) has given one set of semantics that may be applied to a BPL and its specializations in terms of a "dialogue" or "game" between two opponents. Watanabe (1969) has developed a similar logic in terms of a valuation on a lattice commencing with specific semantics in terms of observations of events. This section puts forward a related semantics which provides a common interpretation of all the logics discussed in terms of the responses of population of entities that may, for the sake of intuition, be considered to be people, neurons, or some other element on which binary decisions may be based.

Consider a population each member of which can "respond" to certain questions with a binary, yes or no, reply. The forms of question will involve evaluating a statement which belongs to the generating set, X, of a lattice, L, as defined in Sect. 3. For example, "is this statement, $x \in X$, true or false, or reasonable or unreasonable, or generally believed, etc." The valuation of x is defined to be the proportion of the population replying yes to the question. A compound statement in L is given a valuation in terms of the proportion of the population who say yes to each of x and y for terms of the form, $x \wedge y$, or who say yes to either x or y for terms of the form, $x \vee y$, and similarly for more complex combinations of conjunction and disjunction.

This is essentially a set-theoretic model for L as a (distributive) lattice of subsets of the population, and postulates P1 through P9 are clearly valid. A distance measure, and hence valuations of logical equivalence, implication, and negation, may be defined as in P10 through P14. Thus, for any given population whose members are able to give one of two responses to a question about each element of X, there is a simple and well-defined procedure for ascertaining the valuation of any arbitrary statement in L, involving conjunction, disjunction, equivalence, implication, and negation, which is consistent with P1 through P15. Thus such a population, or a set of such populations, is a model for a distributive SUL.

Consider now the additional constraints that must be placed upon the behavior of individuals within the population if the specializations for this basic logic analyzed previously are to be obtained. Note first that no constraints have been implied so far except the ability to answer a question about a member of X with a yes or no reply. If we also assume that members of the population are able to deal similarly with other statements in L other than those in X (i.e., compound statements), then it is necessary to postulate that each member of the population obeys the rules of inference of that fragment of PC concerned with conjunction, disjunction, and implication (Rescher, 1969, p. 333). The implications of the other additional postulates are:

(a) P16 giving LEM and Rescher's probability logic—a member of the population must give opposite responses to a statement and its negation—or, if we do not assume that negation is meaningful in a question, we assume that the response to a question about the negation of an element of X is the opposite of that for a question about X. Dependent on what type of question we ask, for example about the truth of x, or about the reasonableness of x, this may, or may not, be an intuitively appropriate contraint.

(b) Multiplication of the valuations for conjunction giving a "stochastic logic"—here we must assume that the responses of a certain type to a given question are scattered randomly among the population with no relationship between responses to questions about different elements of X. We could again produce this effect externally by chosing a number of different individuals at random to answer each question involved in our evaluating a compound, although it is difficult to see why we should want to do so! Independence of responses to questions within the population, however, might will be an intuitively reasonable hypothesis.

(c) P17 giving a fuzzy logic—this would apply if members of the population each evaluated the questions according to the same criteria but applied a different threshold to the resulting evidence, or "feeling." The member with the lowest threshold would then always respond with a yes answer when any other member did, and so on up the scale of thresholds. This model, so different from that of independence of responses, also has its intuitive attractions. Reason (1969) has shown that the threshold applied by human beings in coming to a binary decision on an essentially analog variable seems to be associated with personality factors and a trait of the individual. If so, human populations would tend to show more a fuzzy, than a stochastic, logic in their decision making. Similarly the concept of uniformity in data processing but varying thresholds of sensitivity is a reasonable one for populations of cells.

Thus there is a simple, intuitively meaningful semantic basis for a SUL that allows the formulation of constraints paralleling those in the formal logic that lead to Rescher's probability logic and its specialization to a truth-functional logic of statistical independence, or lead to the "fuzzy logic" $Ł_{\aleph_1}$. There is a further, independent dimension of semantic variation when we begin to specify what the population actually is. If we take it to be physical events falling into one of two categories (e.g., occurring or not occurring), then the logic is one of physical probability. This is the interpretation that is rejected in the fuzzy reasoning literature—a degree-of-membership of a 6-foot man of 0.5 to the set of *tall men* certainly has no interpretation in terms of physical measurements of his height, e.g., "on 50 % of the occasions when we measure him he is over 6 feet tall!" However, note that, on the other hand, fuzzy logic connectives *can* readily occur with physical events—if, for example, one event is a necessary effect or a necessary cause of another. Thus the rejection is a question of linguistic modeling, *not* one of fuzzy versus probability logics.

An interpretation that does seem consistent with the use of SULs in modeling human linguistic reasoning is that the population is one of people. The 6-foot man is now reckoned by 50 % of the community to be tall. We are using population stereotypes to develop an underlying model of human linguistic behavior—an eminently reasonable approach if the role of that behavior is to communicate in that community! As noted, Reason's (1969) results indicate that some degree of interconnection between responses leading to a fuzzy logic might be expected in these circumstances.

None of these further specializations of the semantics is *necessary*. The abstract SUL and the general semantics provided by Giles, Watanabe, or those here, are complete in themselves. There is no need to interpret fuzzy reasoning in terms of either individual decision making or population stereotypes—many other conceptions are possible. However, the arguments of this section do illustrate the lack of any basic conflict between fuzzy and probability logics in themselves. SUL and all its derivatives apply equally well to physical probabliity, vague reasoning, subjective probability, belief, and so on. The addition of requirements for LEM or strong truth-functionality do lead to different logics, but do so in *all* these various interpretations.

7. Summary and Conclusions

The prime objective of this paper has been to clarify the essential differences between recent developments in logics of uncertainty based on fuzzy sets theory and previous work with probabilistic foundations. At a formal level one may note that Rescher's probability logic (PL) and Lukasiewicz's $Ł_{\aleph_1}$ have a common set of definitions (those of a SUL) for all connectives including implication. To obtain PL one adds the law of the excluded middle. To obtain $Ł_{\aleph_1}$ one adds necessary implication between arbitrary propositions *or* strong truth-functionality. SUL and its derivatives, PL and $Ł_{\aleph_1}$, have a common semantic model in terms of binary responses from a population. It is only when this population is further specified (in greater detail than many of the formal developments require) that the differences in interpretation of "degrees of membership" appear that have been noted in the fuzzy reasoning literature. These may be attributed entirely to differences in interpretation leading to "physical" or "subjective" probability, for example, rather than to differences in the logics themselves.

It has been shown that while a SUL with LEM cannot be strongly truth-functional (values of connectives equationally defined in terms of truth-values regardless of structure of propositions) it can be made weakly truth-functional (values of connectives all completely defined) by an assumption of statistical independence between propositions that have no generating element in common. It is worth noting that various alternative constraints to statistical independence may be added, including the min/max connectives of fuzzy logic (as was done by Sect. 2). In the resultant logic: LEM holds; the logic is truth-functional; and the required connectives hold between propositions that are not structurally related. This form of variant on a SUL seems to satisfy many of the requirements for a fuzzy logic put forward, for example, by Sanford (1975).

In conclusion, one may suggest that in future, rather than debate what is the *right* set of connectives, one should turn the question about and ask what propositions are fuzzily related, which ones are statistically independent, which ones are mutually exclusive, etc., and use these considerations to define *modalities* in a SUL with, or without, LEM. In terms of the population model one might expect any real population to show a variety of forms of connective and the reasons for this variety would clearly throw much light on the structure of the population itself. The assumption of strong truth-functionality would then appear as a computational device simplifying calculations by enabling the structure of propositions to be forgotten once their truth values had been calculated—bounds could be obtained on the degree of approximation involved if the assumption were not reasonable.

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