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Foundations of fuzzy reasoning

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This paper gives an overview of the theory of *fuzzy sets* and *fuzzy reasoning* as proposed and developed by Lotfi Zadeh. In particular it reviews the philosophical and logical antecedents and foundations for this theory and its applications. The problem of *borderline* cases in set theory and the two classical approaches of *precisifying* them out, or admitting them as a *third case*, are discussed, leading to Zadeh's suggestion of *continuous degrees of set membership*. The extension of basic set operations to such *fuzzy sets*, and the relationship to other *multivalued logics* for set theory, are then outlined. The *fuzzification* of mathematical structures leads naturally to the concepts of *fuzzy logics and inference*, and consideration of implication suggests Łukasiewicz infinite-valued logic as a base logic for fuzzy reasoning. The paradoxes of the barber, and of sorites, are then analysed to illustrate fuzzy reasoning in action and lead naturally to Zadeh's theory of *linguistic hedges and truth*. Finally, the logical, model-theoretic and psychological derivations of numeric values in fuzzy reasoning are discussed, and the rationale behind interest in fuzzy reasoning is summarized.

1. Introduction

Models of human reasoning are clearly relevant to a wide variety of subject areas such as sociology, economics, psychology, artificial intelligence and man-machine systems. Broadly there are two types: psychological models of what people actually do; and formal models of what logicians and philosophers feel a rational individual would, or should, do. The main problem with the former is that it is extremely difficult to monitor thought processes—the behaviourist approach is perhaps reasonable with rats but a ridiculously inadequate source of data on man—the introspectionist approach is far more successful [e.g. in analysing human chess strategy (Newell & Simon, 1972)] but the data obtained is still incomplete and may not reflect the actual thought processes involved.

Formal models of reasoning avoid these psychological problems and have the attractions of completeness and mathematical rigour, hopefully proving a normative model for human reasoning. However, despite tremendous technical advances in recent years that have greatly increased the scope of formal logic, particularly modal logic (Snyder, 1971), the applications of formal logic to the imprecise situations of real life are very limited. Some 50 years ago, Bertrand Russell (1923) noted:

> "All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestial life but only to an imagined celestial existence . . . logic takes us nearer to heaven than other studies."

The attempts of logicians to rectify this situation and broaden the scope of logic to cover various real-world problems has been surveyed recently by Haack (1974), and the role of modern developments in philosophical logic in AI has been excellently presented by McCarthy & Hayes (1969). The current paper is concerned with an area of massive recent development not covered by either of these references, that of *fuzzy sets theory* and *approximate reasoning* initiated by Lotfi Zadeh.

It is no coincidence that Zadeh's previous work had been concerned with successively improved refinement in the definitions of such terms as "state" (Zadeh & Desoer, 1963; Zadeh, 1964) and "adaptive" (Zadeh, 1963) in systems engineering. It was dissatisfaction with the decreasing semantic content of such increasingly refined concepts that lead to his (1972*a*) remarks that:

> "In general, complexity and precision bear an inverse relation to one another in the sense that, as the complexity of a problem increases, the possibility of analysing it in precise terms diminishes". "Thus 'fuzzy thinking' may not be deplorable, after all, if it makes possible the solution of problems which are much too complex for precise analysis."

An independent development of the same conclusions in the context of control engineering has recently been given by Aizermann (1975). They are probably valid in most scientific disciplines where the development of formal theory has had sufficient time to expand way beyond the reach of practice—journals of "X-theory" are often renowned for their irrelevance to "X-practice"!

The interaction of over-precision with lengthy chains of reasoning to produce dubious or nonsensical results is not a new phenomenon. It had been noted by Greek philosophers in such paradoxes (Cargile, 1969) as those of *sorites* (the "heap" that remains one even if an object is removed) or *falakros* (the "bald man" who remains so even if he grows one additional hair). In the more modern context of control theory and practice cited above, consider a study in "control engineering". It clearly remains one if we replace the actual plant controlled with a computer model of that plant. It clearly remains so if we consider the plant model as a set of numeric equations. It continues to remain so if we consider the general algebraic form of these equations. And so on—each step in itself is a small enough change that we agree that the content of the paper cannot have crossed a borderline between "control engineering" and "not control engineering". Yet when the final paper appears (called "Residues of contraction mappings in Banach spaces"!), few control engineers will recognize it as belonging to their discipline.

The sorites paradox and its variants may be seen as arising from our introducing artificial precision into naturally vague (but usable) concepts. Ultimately the motivation for such precision seems to come from a requirement for truth itself to be bivalent—"Either it is true that this is a paper on control engineering, or it is false. Which do you assert?" Whilst many may feel that the logic of science requires such bivalency, and the associated precision, there can be little doubt that we reason quite capably in everyday life without it. For example, the classical syllogism:

Socrates is a man.
All men are mortal.
Socrates is mortal.

has no classical counterpart for:

Socrates is very healthy.
Healthy men live a long time.
Socrates will live a very long time.

and yet would we wish to distinguish the validity of these two argument forms in everyday reasoning?

It was both the paradoxes introduced by over-precision, and the loss of powerful argument forms involving imprecise predicates, that led Zadeh to question the direction taken by methodologies of science that reject the *fuzziness* of concepts in natural use and replace them with non-fuzzy scientific *explicata* by a process of *precisiation.* During recent years (see bibliography) he has developed in detail a model for approximate reasoning with vague data. Rather than regard human reasoning processes as themselves "approximating" to some more refined and exact logical process that could be carried out perfectly with mathematical precision, he has suggested that the essence and power of human reasoning is in its capability to grasp and use inexact concepts directly. Zadeh argues that attempts to model, or emulate, it by formal systems of increasing precision will lead to decreasing validity and relevance. Most human reasoning is essentially "shallow" in nature and does not rely upon long chains of inference unsupported by intermediate data—it requires, rather than merely allows, redundancy of data and paths of reasoning—it accepts minor contradictions and contains their effects so that universal inferences may not be derived from their presence.

The insights that these arguments give into the nature of human thought processes and, in particular, to their modelling and replication in the computer, are of major importance to a wide range of theoretical and applied disciplines—particularly to the role of formalism in the epistemology of science. The arguments have become associated with "fuzzy sets theory" (Zadeh, 1965), and this does indeed provide a mathematical foundation for the explication of approximate reasoning. However, it is important to note that Zadeh's analysis of human reasoning processes and his exposition of fuzzy sets theory are not one and the same—indeed they are quite distinct developments that must be separated, at least conceptually, if a full appreciation is to be had of either. As analogies one may conceive that fuzzy sets are to approximate reasoning what Lebesgue integration is to probability theory; what matrix algebra is to systems theory; or what lattice theory is to a propositional calculus; i.e. vital mathematical tools for certain approaches to the theory but not the theory itself.

Figure 1 was compiled from an up-to-date bibliography on fuzzy systems containing some 600 references (Gaines & Kohout, 1977) and demonstrates the epidemic growth of such work in recent years. The practical relevance of these studies is illustrated by such applications as: pattern recognition (Siy & Chen, 1974); clustering (Bezdek, 1974); political geography (Gale, 1975); decision-making (Baas & Kwakernaak, 1976); robot planning (Goguen, 1975; Kling, 1974; LeFaivre, 1974); chromosome classification (Lee, 1975); medical diagnosis (Albin, 1976); engineering design (Becker, 1973); systems modelling (Fellinger, 1974); process control (Mamdani & Assilian, 1975; Mamdani, 1976); social interaction systems (Wenstøp, 1976); and structural semantics (Rieger, 1976).

The best introduction to fuzzy reasoning is undoubtedly the work of Zadeh himself (e.g. Zadeh, 1965, 1973, 1976; Zadeh, Fu, Tanaka & Shimura, 1975; Bellman & Zadeh, 1976) and this paper is intended as an overview rather than an introduction. There are relationships between fuzzy sets theory and studies of the paradoxes of naïve set theory (Chang, 1965; Chihara, 1973; Maydole, 1975); to studies of the concept of "truth" in formal systems and everyday language (Tarski, 1956; Leblanc, 1973; Mackie, 1973; Evans & McDowell, 1976); to previous studies of vagueness (Russell, 1923; Black, 1937; Fine, 1975), inexact measurement (Adams, 1965; Krantz, Luce, Suppes & Twersky, 1971), and the psychology of inexact expression (Sheppard, 1954); to the work of the Polish logic school of 1920–1939 (McCall, 1967) devastated by the war; to many aspects of work on multivalued logics (Rescher, 1969) where fuzzy reasoning provides a new semantics; to work on modal logics of entailment (Anderson & Belnap, 1975), possibility and necessity (Hughes & Creswell, 1968; Snyder, 1971), quantity (Altham, 1971), knowledge and belief (Hintikka, 1962), and time (Prior, 1967); to the many aspects of probability theory (Fine, 1973) particularly that concerned with "subjective" foundations (Savage, 1971; Finetti, 1972) and belief (Grofman & Hyman, 1973); to studies of the foundations of science (Lakatos & Musgrave, 1970; Gellner, 1974; Hesse, 1974; Feyerabend, 1975) particularly inductive reasoning (Katz, 1962; Kyburg, 1970; Hintikka & Suppes, 1970); and to studies of human linguistics, reasoning and rhetoric (Fillmore & Langendoen, 1971; Creswell, 1973; Hockney, Harper & Freed, 1975; Simons, 1976).

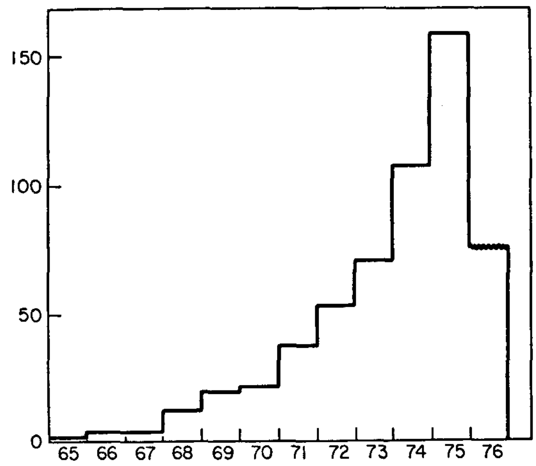

FIG. 1. Histogram of papers on fuzzy systems against year of publication.

In the following sections I shall give a brief outline of fuzzy set theory, fuzzy reasoning, and their relationship to these other areas of study, and attempt to give some feeling for why the subject area has developed so rapidly at this time and where it is going. Section 2 introduces the problem of *borderline* cases in set theory and the two classical approaches of *precisifying* them out of or admitting them as a *third case*. The problems of either approach leading to Zadeh's suggestion of continuous degrees of set membership, and the Kalman–Zadeh debate, are then discussed. Section 3 extends the basic set operations to continuous degrees of membership, presenting the fundamental Bellman and Giertz results. Section 4 introduces other derivations of multivalued logical foundations for set theory and variants on Zadeh's operations. Section 5 describes the *fuzzification* of mathematical structures and illustrates this with the propositional calculus. Section 6 develops the concepts of *fuzzy logics and inference*, paying particular attention to *implication*, and giving an axiom schema for Ł$_1$† as a base logic for fuzzy reasoning. Section 7 illustrates some basic aspects of fuzzy reasoning by showing the different ways

†Ł$_1$ is used throughout as an abreviation for Ł$_{\aleph_1}$, the infinite valued form of Łukasiewicz logic (Rescher, 1969).

in which the paradoxes of the barber, and of sorites, are resolved in fuzzy logic. Section 8 develops Zadeh's theory of *hedges* and *truth* and gives examples of tautologies in fuzzy reasoning involving these concepts, and of non-tautologous fuzzy reasoning. Section 9 is concerned with the role of numerical degrees of membership, and truth values, in fuzzy reasoning, and demonstrate how they may be derived from; the logic in axiomatic form; various intuitively meaningful models; and psychological studies. Section 10 concludes this paper, and is particularly concerned with the reasons for current interest in human linguistic reasoning with imprecise concepts.

2. Fuzzy sets theory

Set theory forms the foundations of arithmetic, logic, and indeed the major part of mathematics and formal reasoning. We tend to move naturally from the classifications of everyday language to the mathematical formulation of a set. A person may be tall—consider the set of tall people; an object may be red—consider the set of red objects; a system may be stable—consider the set of stable systems. The move from a predicate to a set satisfying it, from an intension to an extension (Carnap, 1947), is a powerful tool in mathematics (Wiener, 1914) and a powerful, albeit dangerous, procedure in everyday reasoning (Korzybski, 1958).

However, in all three examples given above the step from predicate to set is a dubious one. Membership of a set is a very precise concept—either a potential element is a member, or it is not. Whilst there may be many people, objects and systems that we can unreservedly declare tall or not tall, red or not red, stable or not stable, respectively, do we actually possess a decision procedure that enables us to classify in this binary fashion any appropriate entity? If we do not, then what is the status of the unclassified entities, the *borderline cases* (Machina, 1972).

One way out of the dilemma is to have no borderline cases. Carnap (1950) puts forward a process of "precisiation" in which everyday concepts are given precise scientific explicata which do not necessarily coincide with their explicanda but which are to be, firstly:

> "Similar . . . in such a way that, in most cases in which the explicandum has so far been used, the explicatum can be used; however, close similarity is not required, and considerable differences are permitted"

and secondly:

> "useful for the formulation of many universal statements".

This process of precisiation has always been an important component in the development of a "science", but it is universal applicability and utility that Zadeh, for one, now questions. Note, however, that Carnap's formulation is quite deliberately explicit about the deviation of the explicandum, justifying this cost in terms of the benefit of being able to obtain universal laws. Thus the criticism of classic methodologies quoted from Russell and Zadeh in section 1—more crudely, that science does indeed, "say more and more about less and less"—is seen by Carnap not as a source of contradiction but rather as a trade-off. There is a continuous spectrum with isolated, but "real", phenomena at one extreme and universal, but "unreal", laws at the other.

However, if one takes "precisiation" in its narrow sense to be a process of explicating out the borderline cases, then there is an alternative approach that has its attractions and that is to treat them separately as a distinct class. Each entity is regarded as a member, a non-member, or a borderline member, of a set. We have a ternary rather than a binary distinction, rather like that of future contingents in logic where an event may be, as yet, neither true nor false but has to be ascribed a third truth value, "possible" (Łukasiewicz, 1930). In developing arguments we can concentrate on the definite cases and leave borderline exemplars outside the debate, e.g. the universal law that, "all birds can fly" is not falsified by the ostrich which is "not quite a bird", a borderline case. There is no doubt that we use this kind of logic in everyday reasoning and are prepared to claim that, "the exception proves the rule". However, it still leaves open the question of where the borderline actually is, giving rise to the secondary phenomenon of entities that are borderline "borderline" cases, and so on!

It is between the thesis of no borderline cases and the antithesis of definite borderline cases that Zadeh (1965) creates the dialectical synthesis of continuously graded degree of membership to a set. This is a natural generalization of the characteristic function, $A : S \rightarrow \{0,1\}$, of a subset, A, of a set, S. For any $\mathbf{x} \in S$, $A\mathbf{x}=1$ if $\mathbf{x} \in A \subset S$, whereas $A\mathbf{x}=0$ otherwise.† Zadeh (1965) suggests that $A\mathbf{x}$ be not restricted to the binary endpoints of the interval, [0,1], but instead be allowed to range continuously throughout the interval. The semantics of intermediate values of $A\mathbf{x}$ are not tightly defined but are to be consistent with the natural order relation on the unit interval, e.g. that $A\mathbf{x}=0{\cdot}6$ denotes a greater degree of membership of $\mathbf{x}$ to A than does $A\mathbf{x}=0{\cdot}4$. Sets with such a graded characteristic function Zadeh calls *fuzzy sets* and proposes that such predicates as tall, red, and stable, do not define classical sets with a binary membership function but instead fuzzy sets with graded membership.

The concept of a fuzzy set has advantages relative to either of the possible approaches to borderline cases so far considered. The deviation that Carnap saw as necessarily introduced in replacing the explicandum by the explicatum is minimized because no artificial precision need be introduced to avoid borderline cases—we do not have to say that, "a tall man has a height greater than 1·82755 m", or that, "a red object is one indistinguishable in hue from a uniformly reflecting surface illuminated with monochromatic light of wavelength between 580·27 and 702·35 nm", and so on. As height decreases so does degree of membership to the (fuzzy) set of tall men—as the wavelength of the matching hue decreases so does degree of membership to the (fuzzy) set of red objects.

On the other hand, we do not have to introduce a clearcut distinction between borderline and definite membership—this would involve the same type of arbitrary numeric threshold but now at the meta-level of degrees of membership. It is interesting to compare Łukasiewicz's (1930) logic of future contingents in which he introduces a third truth

†In fuzzy set theory and logic there is a danger of typographical obscurity with elementary concepts being expressed through a mixture of subscripts and parentheses: $\mu_s(x)$ for the degree of membership of the fuzzy variable x to the fuzzy set s. When the designation of s is itself a complex expression then the typography becomes very messy, e.g. $\mu_{((p \cup q) \cap t)}(x)$, etc. In this paper the convention is adopted that fuzzy variables are in a bold typeface, and that the set to which they are a member is placed to their left to indicate degree of membership, i.e. if **x** is a fuzzy variable belonging to fuzzy set s, then s**x** is the degree of membership of **x** to s. In addition parentheses are dropped whenever there is no ambiguity, e.g. $(p \cup q)\mathbf{x}$ is written $pq\cup\mathbf{x}$ since the expression may be resolved only in one way. Concatenation of letters in the same typeface is taken to bind them into a single symbol, so that "sleek" is the name of a fuzzy set *not* the result of implied operations on five variables. Degrees of membership will always be assumed at least partially ordered so that expressions such as $s\mathbf{x} \leqslant p\mathbf{x}$ may be used without further definition. If arithmetic operations are used on them then a mapping into the unit interval is being assumed. Finally, for **MVLs** the italic form of a logical variable will be used for its truth value, e.g. x is the truth value of x.

value, 1/2 (possible), between 0 (false) and 1 (true). One could do this for set theory with 1/2 being the value of the characteristic function for a borderline element. Zadeh can then be seen to have extended the ternary membership values of a "borderline" set theory, {0, 0·5, 1} to an infinite range of values.

Thus the concept of a "fuzzy set" may be seen as providing a new tool, more appropriate than that of classical set theory, for a programme of precisiation. It allows the inherent imprecision of the concepts that we actually use, and wish to use, to be neither discarded nor introduced explicitly in the explicatum, but rather to be subsumed in the (universal) concept of a degree of membership to a fuzzy set centred on that explicatum. There is thus no conflict between either the objectives of the methodology of Carnap's approach and that of Zadeh—instead, the latter may be seen as a logical extension of the former.

This is an important point to make because there has been misunderstanding of the objectives of work on "fuzzy" systems theory—perhaps, that normal standards of scientific method are to be dropped, or at least relaxed. Kalman in the discussion following Zadeh (1972*b*) states:

> "The most serious objection of 'fuzzification' of system analysis is that lack of methods of system analysis is not the principal scientific problem in the 'systems' field. That problem is one of developing basic concepts and deep insight into the nature of 'systems', perhaps trying to find something akin to the 'laws' of Newton."

This division of opinion is of interest because Kalman and Zadeh have in the past adopted very similar approaches to systems theory, both with great, and related, success. As noted previously, a key feature of this work has been the definition with ever increasing precision of terms such as "controllable", "stable", "adaptive", etc. (Zadeh, 1964; Kalman, Falb & Arbib, 1969). Zadeh now feels that new tools and methodologies are necessary for the furtherance of this work. At least part of the reason for this may be seen in the explosive growth of definitions of apparently simple concepts, such as stability, as system theory attempts to cope with the more subtle features of complex systems. For example, Habets & Peifer (1973) report some 184320 [*sic!*] formally different concepts of stability based on variants of those in the literature, and Gaines (1972, 1974) reports a wide variety of concepts of adaptivity arising out of Zadeh's (1963) original definition. The ultimate precisiation is to treat every event in the world as different from every other—which it is!

Kalman feels that the classical tools and methodologies are adequate, or at least have not proved to be inadequate, for the continued pursuit of "universal" laws of system theory. It would clearly be unfruitful to deliberately "fuzzify" a situation unnecessarily, and there is also the obvious danger of becoming engrossed in a fascinating methodology that has little relevance to the problems it purports to solve. Conversely, however, it would seem equally misleading to pretend to clarify, and to develop methodologies dependent on this pretence, in situations where clarity is inherently impossible except at the expense of losing the very nature of the concept to be clarified. The debate is not about the intrinsic value of fuzzy reasoning, but about whether the imprecision of our knowledge of the real world is inherent, and an essential component of any theory, or whether it can be removed by continuing effort. That is, Kalman is affirming the basic doctrine of the theology of science—if we have no clear and precise model of a phenomenon then we should not rest content with our imperfect knowledge but must continue to search for a better model and continue to have faith that it is there. He quotes Hilbert's

"*Wir wollen wissen: wir werden wissen*".

A continuing search for ever-increasing precision does not in itself appear to be a bad thing. It would certainly seem reasonable to suppose that it is undecidable when we have achieved the maximum possible precision. However, a similar debate has taken place about the role of randomness in system theory (Witten & Gaines, 1976). Many eminent scientists, including Darwin, Freud and Einstein, have regarded randomness as a sign of our ignorance rather than as a phenomenon in its own right. As Hume (1739) notes:

"*it is commonly allowed by philosophers that what the vulgar will call chance is nothing but a secret and concealed cause*"

and this attitude to randomness persists today as is shown by Suppes (1974) vehement attack on the "new theology" that holds such tenets as "every event must have a sufficient determinant cause".

It might be supposed that to hold such a view could be erroneous but that, in itself, it cannot do any harm, and, as Kalman has suggested, it is important to continue to act as if there were precise universal laws. However, Gaines (1976*a*) has shown that a modeller assuming causality faced with a sequence generated by a system having the slightest acausal component forms a model which is not just incorrect but totally meaningless. Whereas a probabilistic modeller (Gaines, 1976*d*, *e*) can acquire an accurate, if uncertain, model of the actual system generating the behaviour. Thus it is dangerous to assume certain forms of precision when they do not exist in the world. Interesting results may be obtained but they stem from the mismatch between presupposition and observation, not from the nature of the observed system itself.

Comparable examples in terms of fuzzy, rather than probabilistic, uncertainty may be found in such statements as "*the number of trees in Canada is even*", which was used by Putnam (1976) as an example of a sentence to which Tarski's criterion of truth cannot be applied. The statement is well formed in all the obvious ways and looks superficially open to an operational decision procedure for empirical verification. However, on deeper inspection of the requirements the effect of the fuzziness of both the Canadian boundary and the nature of a tree upon the determination of evenness becomes apparent, and the sentence appears nonsensical. Indeed one can see that "evenness" of a quantity is more precise than the concept of a quantity itself—large numbers become fuzzily represented in everyday language, and the exact cardinality required to determine evenness would be unusual.

Much has been written about truth and vagueness and it is impossible to review all of it here. To round off this survey it seems appropriate to give the last word to Karl Popper whose "unended quest" has been for a theory of knowledge that mirrors "reality". In his autobiography (1976) he notes:

"both precision and certainty are false ideals. They are impossible to attain, and therefore dangerously misleading if they are uncritically accepted as guides. The quest for precision is analogous to the quest for certainty, and both should be abandoned. I do not suggest, of course, that an increase in the precision of, say, a prediction, or even a formulation, may not sometimes be highly desirable. What I do suggest is that it is always undesirable to make an effort to increase precision for its own sake—especially linguistic precision—since this usually leads to lack of clarity, and to a waste of time and effort on preliminaries which often turn out to be useless, because they are bypassed by the real advance of the subject: one should never try to be more precise than the problem situation demands."

The key point in Popper's argument is that linguistic precision should grow *ad hoc* to meet the demands of the "problem to be solved". *Clarity* of explanation stems from the naturalness and simplicity of the concepts and vocabulary used. The complexity of additional precision should only be introduced when, of necessity, we are forced to "make new distinctions—*ad hoc*, for the purpose in hand". This emphasis on the primacy of natural, "fuzzy" concepts, and the need to justify and motivate increased (linguistic) precision seems to epitomize the attitudes of those attracted to Zadeh's fuzzy set theory. The distinction that Popper draws between *linguistic* precision and *problem* precision seems also to point to a source of confusion between proponents and opponents of fuzzy reasoning. It may well be a meaningless distinction in certain approaches to epistemology, but it is one that will make sense to the majority of practising scientists and engineers.

Thus the two contrasting points of view arising out of the Kalman–Zadeh debate both have their merits and their dangers. If we assume and accept fuzziness in the world to cloak our own ignorance then we may never make the necessary effort to discern possible precise underlying phenomena. If, however, we refuse to allow that certain concepts are inherently imprecise and yet still useful then we may generate a multitude of alternatives of increasing precision yet decreasing application. Fuzzy set theory in itself is neutral since it allows for "crisp", or precise, concepts as well as fuzzy ones. The only advantages and dangers are in the way that we use it, and the applications studies already noted are beginning to demonstrate that it can be used effectively.

3. Operations on fuzzy sets

Whilst the concept of a characteristic function allowing the degree of membership of an element to a set to range continuously through the interval, [0,1] is itself appealing in developing explicata for certain concepts, it requires further extension if a fuzzy set thus defined is to assume a role comparable to that of a classical set. What do we mean by the complement of a fuzzy set, or by the union, or intersection of two fuzzy sets? If a car, **x**, belongs 0·7 to the (fuzzy) set of sleek cars (we shall write sleek**x**=0·7) and 0·9 to the set of fast cars (fast**x**=0·9), then what degree of membership to the sets of not-sleek, sleek-or-fast, or sleek-and-fast cars does it have? These questions become particularly interesting when we have defined new concepts in linguistic terms as combinations of the others, e.g. swish=(sleek and fast).

One legitimate answer would be that we cannot tell—the degrees of membership, A**x**, B**x**, of an element, **x**, to sets, A, B, may not be sufficient information to determine its membership to the complements, $\bar{A}$, $\bar{B}$, the union $A \cup B$, or intersection, $A \cap B$. For example, we may need to know more of the structures of A and B and their relationships, and even this may not be enough. However, in any theory of fuzzy set operations that is to reduce to classical set theory when degrees of membership are restricted to the binary values, {0,1}, we must have at least the constraints:

$$A\mathbf{x}=0 \rightarrow \bar{A}\mathbf{x}=1; \tag{1}$$

$$A\mathbf{x}=1 \rightarrow \bar{A}\mathbf{x}=0; \tag{2}$$

$$A\mathbf{x}=0,\ B\mathbf{x}=0 \rightarrow A \cup B\mathbf{x}=0,\ A \cap B\mathbf{x}=0; \tag{3}$$

$$Ax=0,\ Bx=1 \rightarrow A\cup Bx=1,\ A\cap Bx=0; \tag{4}$$

$$Ax=1,\ Bx=1 \rightarrow A\cup Bx=1,\ A\cap Bx=1. \tag{5}$$

There are further constraints if the natural numerical order relation of degrees of membership is to be consistent with our concepts of union and intersection. We must have that the degree of membership in the union of A and B (member of either) is not less than membership in either:

$A\cup Bx \geqslant Ax$ and $A\cup Bx \geqslant Bx$, which may be expressed:

$$A\cup Bx \geqslant \max(Ax,Bx) \tag{6}$$

and degree of membership in the intersection of A and B (member of both) is not more than membership in either:

$A\cap Bx \leqslant Ax$ and $A\cap Bx \leqslant Bx$, which may be expressed:

$$A\cap Bx \leqslant \min(Ax,Bx). \tag{7}$$

Similarly, for consistency between the semantics of a small change in degree of membership and a small numerical change in characteristic function, it is reasonable to require that as Ax increases continuously from 0 to 1, then $A\cup Bx$ and $A\cap Bx$ should, for constant Bx, neither decrease nor jump discontinuously, i.e.

$$A\cup Bx,\ A\cap Bx \text{ are continuous, non-decreasing in } Ax,\ Bx. \tag{8}$$

In addition, for consistency with the algebraic framework of ordinary set theory, one may require the normal constraints of associativity, communitativity, idempotency and distributivity on the union and the intersection:

$$(A\cup B)\cup Cx=A\cup(B\cup C)x,\ (A\cap B)\cap Cx=A\cap(B\cap C)x; \tag{9}$$

$$A\cup Bx=B\cup Ax,\ A\cap Bx=B\cap Ax; \tag{10}$$

$$A\cup Ax=Ax,\ A\cap Ax=Ax; \tag{11}$$

$$(A\cup B)\cap Cx=(A\cap C)\cup(B\cap C)x,\ (A\cap B)\cup Cx=(A\cup C)\cap(B\cup C)x. \tag{12}$$

In a classic paper, Bellman & Giertz (1973) show that (3) through (12) are consistent with one another and sufficient to constrain the inequalities of (6) and (7) to be equalities, i.e.:

$$A\cup Bx=\max(Ax,Bx),\ A\cap Bx=\min(Ax,Bx). \tag{13}$$

Since all we have required is existence, continuity, semantic consistency with the natural order on the unit interval, and algebraic consistency with standard set theory, this result shows that the use of a max function to compute the union of two fuzzy sets and of a min function to compute the intersection of two fuzzy sets [as proposed by Zadeh (1965) in his original paper] is a standard extension of set theory and, in some sense, natural and necessary if the characteristic function of a set is to be extended to range throughout the interval [0,1].

Figure 2 illustrates how the standard and fuzzy set unions and intersections relate to one another: the top diagram shows two standard sets, A and B, intersecting and gives the values of the characteristic function for A, B, $A\cup B$, $A\cap B$, in various regions; the four plots below show how the characteristic functions vary along the line drawn through the sets; the four plots below this show the same variations for fuzzy sets, A′ and B′ similar to A and B but with a graded characteristic function. Clearly the ellipses delimiting the boundaries of the non-fuzzy sets, A and B, would have to be replaced by mounds coming out of the paper in order to show the fuzzy sets, A′ and B′, in a comparable fashion.

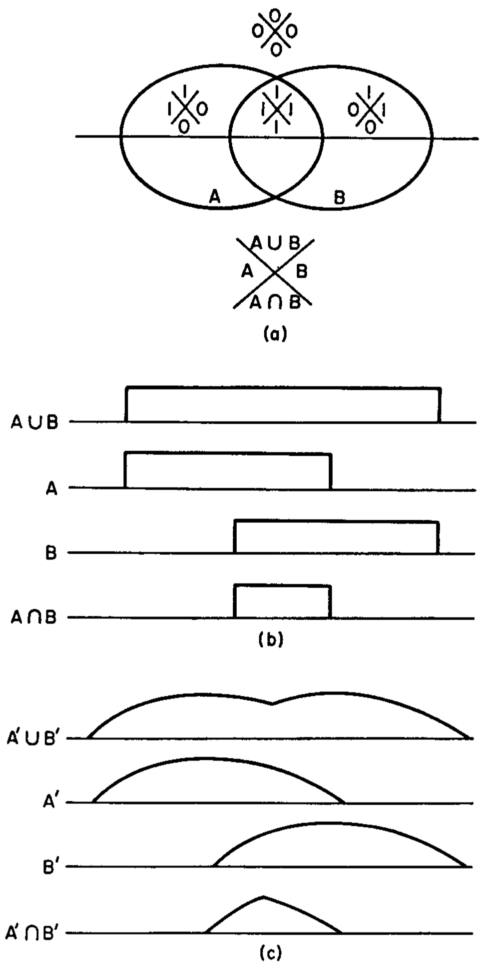

FIG. 2. Sets, fuzzy sets and operations. (a) Binary characteristic function of two sets, A and B, their union and intersection. (b) Variation of characteristic functions along centre line for A, B, classical sets. (c) Possible variation of characteristic functions along centre line for A′, B′, fuzzy sets.

What of complementation? Again the semantics of the numerical order may be considered and clearly require that complementation be order-reversing:

$$Ax > Bx \rightarrow \bar{A}x < \bar{B}x. \tag{14}$$

Bellman & Giertz (1973) suggest that in addition to this order reversal complementation should be such that:

$$\bar{\bar{A}}x = Ax. \tag{15}$$

so that the complement of the complement of a fuzzy set is the original set. But, as they point out, these two constraints are still insufficient to determine a unique function for complementation. However, it is readily shown that if we add a symmetry requirement, essentially that the effect of complementation on a deviation of membership function from unity is the same as on a deviation from zero, so that:

$$Ay=1-Ax \rightarrow \bar{A}y=1-\bar{A}x \quad (16)$$

then (14), (15) and (16) taken together imply:

$$\bar{A}x=1-Ax \quad (17)$$

which is Zadeh's original definition for the membership function of the complement of a fuzzy set.

The computations of the membership function of an element in the complement of a fuzzy set, or in the union or intersection of two fuzzy sets, given by (17) and (13), are by far the most widely used in the literature both in theoretical and applied studies. Complementation is the most weakly constrained and subject to variation, but, as I shall illustrate in section 5, it is not used in some important developments such as the "fuzzification" of other mathematical structures.

4. Some alternative formulations

Whilst Zadeh's fuzzy sets theory arose from the semantics of applied systems analysis, there have been developments of a rather more formal nature that closely parallel it (Rescher, 1969, p. 13). The delightful story of Russell's discovery of a paradox in Frege's *Grundsetze der Arithmetik*, and Frege's reactions to it, are well-known (for translations of the original correspondence see Heijenoort, 1967, pp. 124–128). Russell attempted to resolve it through his "theory of types" which, together with the original paradox and its variants (Kleene, 1952), has been a massive stimulation to a variety of developments in mathematics ever since (Heijenoort, 1967). One of these has been the application of multivalued logic (**MVL**) to set theory.

Russell's original version of the paradox (Chihara, 1973) depends on the principle of bivalence in the form of the "law of the excluded middle" (**LEM**) that p and ~p cannot both be asserted. Shaw-Kwei (1954) noted that certain multivalued logics, notably Łukasiewicz Ł_3, in which **LEM** does not hold do not show this paradox, nor the variant of it not involving negation developed by Curry (1942). However, he also demonstrated that other forms of inconsistency leading to paradoxes can still arise.

These paradoxes arise from the axiom schema of "comprehension" in naïve set theory (here abbreviated to **COM**), by which every "well-defined property" determines a set. It has the general form:

$$\forall\, z_1, \ldots, z_n, x, y, x \in y \Leftrightarrow P(x, z_1, \ldots, z_n)$$

where P is any property-defining expression. Skolem (1957) noted that **COM** may be inconsistent if P is an expression in a bivalent logic, even if quantifier-free, whereas it is then consistent in the infinite-valued Łukasiewicz logic, Ł_1 [and Ł_3 (Skolem, 1960)]. Chang (1963, 1965) and Fenstad (1964) extended this result to allow for various restricted forms of quantification in P.

Recently Maydole (1975) has tidied up and extended these developments using a concise methodology for proving **COM** inconsistent for arbitrary logics underlying P.

He demonstrates that the paradoxes that arise will also arise with quantified versions of various well-established non-standard logics, and probability logic. Only a few infinite-valued logics (including those of Łukasiewicz and Post) cannot be shown to lead to inconsistency in **COM** by Maydole's technique, and he argues that it is plausible that consistent versions of naïve set theory could be developed based on them.†

That work on the paradoxes of naïve set theory, and that on fuzzy sets theory, should result in a common advocation of a switch to underlying multivalued logic, is not surprising in retrospect. Both the paradoxes of Russell *et al.*, and those of borderline cases, arise from bivalency, or more subtle logical constraints, on P, the property-defining predicate. The standard counter-example for non-fuzzy theories of vague predicates is the sorites paradox and this involves a long chain of iteration closely resembling those generated in Maydole's method. Thus, there are close links between Zadeh's very practical arguments in terms of the applications of systems theory, and the apparently far more abstract and fundamental studies of set theory and the foundations of arithmetic.

A difficulty in fuzzy sets theory that becomes a major problem in multivalued logical foundations for set theory is that "the numbers are not available early enough", i.e. set theory is a necessary foundation to the arithmetic that allows us to talk in terms of a "degree of membership" of 0·3. Formally, this may be overcome by using an axiomatic form of the multivalued logic rather than introducing it in terms of truth-values. However, there have also been some recent studies of more fundamental derivations. Varela (1975) has extended Brown's (1969) calculus of "distinctions" to include the paradoxical cases as a basis for an analysis of "self-reference". Chapin (1974, 1975) is developing a "set-valued set theory" in which the degree of membership of one "set" to another is itself a "set", and is axiomatizing this through to arithmetic in close correspondence with the classical Zermelo-Fraenkel theory. Goguen (1974*a*) has established a category-theoretic framework for fuzzy sets theory, and has linked the axioms for the category very closely to a phenomenological analysis of human "concepts".

At a more specific level, it is possible to accept Zadeh's extension of the characteristic function to range over the whole unit interval, to require that the theory reduce to the standard set theory in the 0/1 case, and to accept many of the constraints defined in the previous section, but, by rejecting others, to generate different functions for the union, etc. Various authors have suggested alternative functions, often within the framework of fuzzy logic rather than fuzzy sets theory, but it is convenient to consider them at this stage and relate the alternative functions to the constraints dropped.

For example, Zadeh (1976, Appendix) suggests the functions:

$$A \cup Bx = Ax + Bx - Ax \times Bx, \tag{18}$$

$$A \cap Bx = Ax \times Bx, \tag{19}$$

somewhat akin to the rules for the calculation of the measures of unions and intersections of Borel sets of independent events in probability theory. These satisfy (1) through (5) reducing to classical results in the binary case. They also satisfy (6) through (8) preserving the order semantics and (9) through (10) preserving associativity and commutativity. However, neither function gives the idempotency of (11) and taken together, they do not give the distributivity of (12).

†Maydole's (1972) doctoral dissertation can be warmly recommended as a comprehensive introduction to the role of MVLs in the foundations of set theory. His later paper is very much condensed and the full thesis is more readily assimilated.

Despite their algebraic weakness these functions, together with the complementation of (17), have been suggested as more appropriate in their semantics than the max/min operations of (13) by several authors. Koczy & Hajnal (1975) develop them axiomatically from semantic considerations that include a deliberate exclusion of idempotency:

> "The repeated reference to a statement in a disjunctive or conjunctive connection . . . causes an increase and decrease of the acceptance, respectively".

Giles (1976) has also argued that lack of idempotency may be desirable in a logic reflecting the semantics of natural language. Goguen (1969) uses (19) on a similar basis that a chain of links in an argument, each with the same degree of membership to "truth" should overall have a lesser degree of truth that decreases with the length of the chain. Rödder (1975) presents data from human decision-making indicating that this predicted decrease does actually occur [it would be expected to do so on a Bayesian model of information acquisition as investigated by Edwards *et al.* (1968)]. Hamacher (1976) has followed this result by developing a two-paramenter variant of (18) and (19) based on Bellman & Giertz's arguments but dropping idempotency and distributivity, and strengthening (8) to require that union and intersection be strictly increasing functions of their arguments.

In terms of the chain analogy the min/max connectives may be viewed as assuming that a chain of elements in series is as strong as its weakest link, whilst a set of elements in parallel is as strong as the strongest. These concepts are also intuitively appealing and the debate on the "correct" forms of function for fuzzy unions and intersections is likely to continue with different outcomes for different semantic constraints and intuitions. Gaines (1975) has compared the functions of (18) and (19) with those of (13) in a re-analysis of the very successful fuzzy linguistic controller of Mamdani & Assilian (1975) and reports no significant variation in the overall control policy with the different forms of fuzzy connective. It may well be that in real-world applications where there is essential redundancy and robustness in the problem formulation that the precise form of fuzzy function does not matter, i.e. the operations may be fuzzy as well as the data.

Sanford (1975), again in the context of vague reasoning, presents various intuitive arguments that rule out the simple max/min functions and goes on to develop his own "borderline" logic. This has the property, unique among current variants, of dropping the continuity requirement of (8). For example, when a and b are two atomic propositions such that $a+b<1$ then, if $c=a \vee b$, $c=\max(a,b)$, but otherwise it is 1. Thus, if $a=0{\cdot}49$, $b=0{\cdot}50$, then $c=0{\cdot}50$, but if $a=0{\cdot}51$, $b=0{\cdot}50$, then $c=1$, a discontinuous jump. Sanford's paper is particularly interesting both for its innovative approach and for its illustration of the variety of phenomena that can arise in logics of vagueness.

Gaines (1976*b*, *c*) has attempted to integrate together many variants of logics of vagueness, including those based on min/max connectives, probability logic and (18)/(19) above, by dropping the functionality of the union and intersection. He requires only that:

$$A \cup Bx + A \cap Bx = Ax + Bx \tag{20}$$

plus the algebraic constraint that $\cup$ and $\cap$ act as lattice operators in expressions. This gives a common foundation for many variants of fuzzy sets theory and fuzzy logics from which many key results may be derived. Particular variants then arise through additional constraints which may be based on specific semantic requirements. The min/max function proposed by Zadeh assumes the key role, as suggested by the Bellman

& Giertz result, that they are uniquely *strongly* truth-functional, i.e. to ascertain the truth values of connectives between compound propositions it is necessary to know only the truth values of these propositions, not their structures in terms of atomic propositions. This strong truth-functionality substantially reduces the memory requirements of an information processing system, and would be an advantage to a limited capacity decision maker such as the human being.

Thus there is rich ground underlying the particular choice of non-bivalent logic with which to extend classical set theory. The framework established by Bellman & Giertz is useful in comparing variants. However, only the semantics of particular applications can determine which is the appropriate choice. In many practical situations these semantics are themselves so "fuzzy" that the choice does not matter over a wide range of possibilities. Developments of the mathematical foundations of vague reasoning need to take this into account and *not* attempt to introduce a new level of arbitrary precision in the metalanguage—detailed and specific arguments as to what are the "right" functions seem singularly inappropriate to the subject area. We need integrative, broadly-based theories with strong intuitive appeal.

5. Fuzzification of mathematical structures

Given the concept of a fuzzy set and the definition of union and intersection in (13), it is possible to *fuzzify* any domain of mathematical reasoning based on set theory. The fundamental change is to replace the precise concept that a variable has a *value* with the fuzzy concept that a variable has a *degree of membership to each possible value*, i.e. instead of having a sharp value each variable is fuzzily restricted to a domain of values. Sharp values then consist of one particular form of restriction, i.e. a singleton having degree of membership unity with all other degrees zero. A conventional "set-valued" variable consists of a number of values with degree of membership unity and all other degrees zero. Other forms of restriction may be imposed to constrain the relationship between the fuzzy structure and additional mathematical structures, e.g. that the fuzzy set is *normalized* in the sense that at least one value has degree of membership unity.

In terms of the notation of section 3 it is natural to write $v\mathbf{x}$ for the degree of membership of the fuzzy variable $\mathbf{x}$ to the value v, i.e. values themselves now appear to play the role of functions. Hence, non-fuzzy functions become *functionals* mapping function to function when fuzzified. Consider a (non-fuzzy) function, g, of n variables, y_1, through y_n. Before fuzzification we write as usual:

$$v=g(y_1,y_2,\ldots,y_n)=g(Y). \tag{21}$$

A natural extension to $v\mathbf{x}$, the degree of membership of the fuzzy variable $\mathbf{x}$ to each value v, is:

$$v\mathbf{x}=\begin{cases}\max\limits_{Y}(\min(y_1\mathbf{x},y_2\mathbf{x},\ldots,y_n\mathbf{x})) \text{ where } v=g(Y),\\ 0, \text{ if there is no } Y \text{ such that } v=g(Y).\end{cases} \tag{22}$$

That is: with every possible value of the argument of the function is associated a degree of membership that is the lowest of the degrees of membership of each of its components; and with each possible value of a result of the function is associated a degree of membership that is the highest of those of all the arguments giving that value. We are taking the union of the fuzzy sets of results arising from the intersection of the fuzzy sets of values of components. Note that fuzzification does *not* involve the complementation of a fuzzy set, which is advantageous since we have noted that this operation is not as well defined as those of union and intersection.

Many fuzzified mathematical structures have now been studied and basic results extended from the original structures to the fuzzy ones replacing them, e.g. fuzzy topological spaces (Chang, 1968, Goguen, 1974*b*; Wong, 1975; Hutton, 1975); groups (Rosenfeld, 1971); graphs (Rosenfeld, 1975); automata (Santos & Wee, 1968; Mizumoto, Toyoda & Tanaka, 1969; Gaines & Kohout, 1975; Wechler, 1975); algorithms (Santos, 1970); languages (Thomason & Marinos, 1974; Honda & Nasu, 1975); and logics (Lee & Chang, 1971; Gaines, 1976*b*, *c*). In the context of fuzzy reasoning the fuzzification of logic is of particular interest and I shall devote the remainder of this section to the application of the formula of (22) to the standard propositional calculus (**PC**).

Any logical structure may be fuzzified by considering propositions not to have a single truth value, but instead to have fuzzy degrees of membership to each possible truth value. **PC**, the conventional propositional calculus, has two truth values, F and T, so that after fuzzification each proposition, **x**, will be represented by a pair of values (F**x**, T**x**). These may be regarded as representing its degree of membership to falsity, and to truth, respectively. The logical connectives of PC are truth-functional and hence may be represented as functions of their arguments and fuzzified. For example, the truth table for implication, $\supset$, is:

if c=a⊃b		a=F	a=T
	b=F	c=T	c=F
	b=T	c=T	c=T

which gives, by application of (22):

$$(F\mathbf{c},T\mathbf{c})=(\min(T\mathbf{a},F\mathbf{b}),\max(\min(F\mathbf{a},F\mathbf{b}),\min(F\mathbf{a},T\mathbf{b}),\min(T\mathbf{a},T\mathbf{b}))). \tag{23}$$

Similar expressions may be derived by fuzzifying the truth tables for negation, $\sim$, disjunction, $\vee$, conjunction, $\wedge$, and equivalence, $\equiv$, but they are more simply obtained by noting that fuzzification preserves the interdefinability of the connectives of **PC**. That is, if f is any false proposition such that (Ff,Tf)=(1,0), then we may write

$$\sim a \text{ for } a \supset f, \tag{24}$$

$$a \vee b \text{ for} \sim a \supset b, \tag{25}$$

$$a \wedge b \text{ for} \sim(\sim a \vee \sim b), \tag{26}$$

$$a \equiv b \text{ for } (a \supset b) \wedge (b \supset a). \tag{27}$$

For example, (24) when substituted in (23) gives us

$$\text{if } b=\sim a \text{ then } (F\mathbf{b},T\mathbf{b})=(T\mathbf{a},F\mathbf{a}) \tag{28}$$

and, similarly, expressions may be derived for the other connectives.

Thus fuzzified **PC** may be shown related to a well-known **MVL** under a simple transformation. If we assume the fuzzy truth values are *normalized* to have only one non-unity component, there is a 1–1 correspondence between them and the unit interval that simplifies the above expressions. Let:

$$a=(1-\mathbf{Fa}+\mathbf{Ta})/2 \tag{29}$$

and so on for the other variables (this transformation can be inverted given that one of **Fa** and **Ta** must be 1). Then the equations for the new variables under the different connectives become:

$$\mathrm{c}=\mathrm{a}\supset\mathrm{b} \rightarrow c=\max(1-a,b); \tag{30}$$

$$\mathrm{b}={\sim}\mathrm{a} \rightarrow b=1-a; \tag{31}$$

$$\mathrm{c}=\mathrm{a}\vee\mathrm{b} \rightarrow c=\max(a,b); \tag{32}$$

$$\mathrm{c}=\mathrm{a}\wedge\mathrm{b} \rightarrow c=\min(a,b); \tag{33}$$

$$\mathrm{c}=\mathrm{a}\equiv\mathrm{b} \rightarrow \mathrm{c}=\min(\max(1-a,b),\max(1-b,a)). \tag{34}$$

This system of equations gives an **MVL** that Rescher (1969, p. 50) calls the infinite valued form of a variant standard sequence (**VSS**). It was first developed by Dienes (1949) who actually obtains it by replacing Łukasiewicz's implication with the material implication of **PC**. Hence it is not too surprising that normalized, fuzzified **PC** turns out to be Dienes' **VSS**. Note that the $1-a$ definition of negation arises naturally from our transformation and was *not* introduced through a fuzzy complement operation.

6. Fuzzy logics and inference

To use fuzzy set theory as a basis for reasoning with imprecise concepts one needs to develop the notion of a *fuzzy logic* in greater depth and determine what are valid forms of inference with such logics. This is the area of much current research and it is worth noting initially that there are differences in terminology in the literature leading to at least three distinct denotations for the term "fuzzy logic".

(a) *A basis for reasoning with vague or imprecise statements.* This very general definition is consistent with the colloquial use of the term "fuzzy" and its use in technical senses different from that of Zadeh (e.g. Poston, 1971; DalCin, 1973), or in more general formulations (e.g. Goguen, 1974*a*; Arbib & Manes, 1975). "Fuzzy" becomes a modern term replacing previous usage in the literature of terms such as "inexact" or "vague". One may give the term a reasonable definition by noting that it is applicable to predicates defining concepts that have no well-defined borderline and are such that "hedges" such as "very" may be applied to them, e.g. "very tall", "very beautiful", but not "very pregnant" or "very dead".† Imprecision gives rise to fuzziness because it blurs the borderline, and vagueness usually has a connotation of excessive fuzziness that makes a definition difficult to use.

(b) *A basis for reasoning with imprecise statements using fuzzy sets theory for the fuzzification of logical structures.* This more restricted form of definition (a) comes closest to being the intensive form of that given extensively by Zadeh's own papers. However, it is only the more recent of these papers that consider fuzzy reasoning as

†I am dubious about the reality of this distinction. As we come to define a concept more closely it is invariably found to be fuzzy. For example, with modern studies of "heart death", "brain death", etc., it is clear that the actual occurrence of death is fuzzy. One can well envisage a clinician stating, "He is almost dead. He is now very dead"!

such and come to use the term "fuzzy logic" (Zadeh, 1975; Bellman & Zadeh, 1976). In earlier papers Zadeh fuzzifies a variety of mathematical structures to provide models of their use in approximate, linguistic reasoning by people, and does not treat linguistic terms denoting truth, such as "very true" any differently from terms denoting other, less abstract concepts, such as "very hot". However, the development of complete logical structure for fuzzy reasoning has been the prime long-term objective (Zadeh, 1972*a*, 1973):

> "much of the logic behind human reasoning is not the traditional two-valued or even multi-valued logic, but a logic with fuzzy truths, fuzzy connectives and fuzzy rules of inference",

and his most recent papers cited above have gone far to develop just such a logic.

(c) *A multivalued logic in which truth values are in the interval, [0,1], and the valuation of a disjunction is the maximum of those of the disjuncts, and that of a conjunction is the minimum of those of the conjuncts.* This restricted definition has been widely used (Marinos, 1969; Lee & Chang, 1971; Lee, 1972) but it applies to most **MVLs** that have been studied in detail (Rescher, 1969) and may be regarded as defining a family of infinite valued **MVLs** differing only in their implication functions. The definition may be generalized to truth values in a lattice (Goguen, 1969; Brown, 1971; DeLuca & Termini, 1972) or ordered semiring (Schützenberger, 1962; Arbib, 1970; Goguen, 1974*a*, Gaines & Kohout, 1975) or specialized to include the truth value of negation as being one minus the truth value of the statement negated. However, all variants of this definition require statements to have truth values in an ordered structure, and define the logical connectives in terms of the order relation.

Clearly definitions (a) and (b) are compatible, differing only in generality. However, there is scope for some confusion, particularly between (b) and (c), because

(i) some "fuzzy logics" in sense (c) have no derivation in sense (b) or application in sense (a);
(ii) however, as shown in section 5, some fuzzified logics in sense (b) are also **MVLs** in sense (c);
(iii) Zadeh fuzzifies in terms of (b) logics which are already fuzzy in sense (c).

In retrospect it would be very much better if the term "fuzzy logic" had been used consistently in the sense of (b), with the term **MVL** used generically for logics satisfying (c) and the standard names applied, such as **VSS**, to those whose implication functions are sufficiently well-specified for their classical counterparts to be determined. Hopefully, the literature will move towards this more consistent terminology in the future.

Given the representation of imprecise concepts in terms of fuzzy sets described in previous sections, what would constitute a logic for fuzzy reasoning? Clearly some definition of *truth* and a basis for inferring the *consequences* of true statements is required. By the very nature of the imprecise concepts involved one cannot expect a clear-cut distinction between a statement being true, and its not being true. It is natural to take the truth value of a statement expressing the membership of a particular individual to a fuzzy set to be the *degree of membership* to that set, i.e. the truth value of "this snow is white" is the degree of membership of *this snow* to the fuzzy set of *white objects*. A valid inference will then be one that takes us from a premise to a consequence of at least equal truth value. Note that both these concepts are consistent with the non-fuzzy case: the truth value of the predicate, P, in an axiom of the form of **COM** (section 4) may be taken as the "degree of membership" of x to y, and the definition of material implication, $\supset$, in **PC** is completely equivalent to saying that $x \supset y$ is true if and only if the truth value of y is at least equal to that of x.

Unfortunately, fuzzified **PC**, or **VSS**, does not have this property for its implication function. For example, substituting $c=1$ in (30) gives us, $a=0$ or $b=1$, and not the simple inference rule, $b \geqslant a$, i.e. **VSS** allows an inference to be precisely true only if the premise is precisely false or the consequent precisely true—it essentially forces us back to a non-fuzzy logic. Many other **MVLs** *do* satisfy the more useful condition that:

$$\text{if } c = a \supset b, \text{ then } c=1 \rightarrow b \geqslant a. \tag{35}$$

Note that the converse does not necessarily hold, though it may do in some **MVLs**.

In terms of Zadeh's exposition of fuzzy reasoning, and in virtually all application of fuzzy reasoning, it seems that the rules for disjunction conjunction, and negation of (32), (33) and (31), respectively, together with the inference role of (35) are an adequate definition of a base **MVL** to be fuzzified. In essence, implication is being used only metalinguistically, e.g. in Mamdani's (1976) fuzzy linguistic controller a *linguistic rule*, such as "*if* pressure error is negative small and the change in pressure error is positive small *then* heater control change should be positive medium" is formalized as, rule_n.

$$\begin{gathered}(\text{condition belongs to condition}_n) \text{ and } (\text{input belongs to input}_n) \text{ implies}\\ (\text{input belongs to actual input})\end{gathered} \tag{36}$$

where "condition_n" is a fuzzy predicate specifying the plant conditions for the rule to be applied ("pressure error negative small and change positive small"), "input_n" is a fuzzy predicate specifying the input constraint called for by the rule ("heater control change positive medium"), "actual input" is a fuzzy predicate restricting the control action to be applied to the plant, and "condition" and "input" are fuzzy variables specifying the actual plant condition and input, respectively. More formally (36) may be expressed:

$$(\mathbf{condition} \in \text{condition}_n) \wedge (\mathbf{input} \in \text{input}_n) \supset (\mathbf{input} \in \text{input}) \tag{37}$$

where the implication expressed by the rule is taken to be precise so that we have, from (35):

$$\text{input } \mathbf{input} \geqslant \min(\text{condition}_n\ \mathbf{condition},\ \text{input}_n\ \mathbf{input}). \tag{38}$$

If there are a number of rules of this type, rule_1 to rule_n, we can infer:

$$\text{input } \mathbf{input} \geqslant \max_n(\min(\text{condition}_n\ \mathbf{condition},\ \text{input}_n\ \mathbf{input})) \tag{39}$$

which is the basis of inference used in the fuzzy linguistic controller (Mamdani & Assilian, 1975).

If implication is to be used in the object language as well as the metalanguage then the truth value of $a \supset b$ itself needs definition. Several are possible that have reasonable interpretations and conform with (35), e.g. if $c = a \supset b$:

$$c = \begin{cases} 1 \text{ if } a \leqslant b, \\ 0 \text{ otherwise;} \end{cases} \tag{40}$$

$$c = \begin{cases} 1 \text{ if } a \leqslant b, \\ b \text{ otherwise;} \end{cases} \tag{41}$$

$$c=\min(1,1-a+b); \tag{42}$$

$$c=\min(1,b/a). \tag{43}$$

Taking these forms of implication with the connectives of (31), (32), (33), and (27), gives (Rescher, 1969) the infinite valued forms of: the sequence S_n (40); the sequence S_n* (41); and Łukasiewicz $Ł_1$ (42). The implication function of (43) does not seem to have been studied in conjunction with the other connectives, but it is closely related to conditional probability. Indeed Gaines (1976*b*, *c*) shows that (42) can be derived from:

$$p(a\supset b)=1-p(a)+p(a\wedge b) \tag{44}$$

and (43) from

$$p(a\twoheadrightarrow b)=p(a\wedge b)/p(a) \tag{45}$$

which are the standard definitions for implication in a probability logic, and a conditional probability logic, respectively.

Zadeh (1975, 1976) has suggested that Łukasiewicz $Ł_1$ be taken as the base logic for fuzzification in modelling linguistic truth values. However, it is worth emphasizing that whilst this is an important possibility, particularly in view of the wealth of study of this particular **MVL**, there is inadequate evidence from either theory or practice as yet to discriminate between the different possibilities given above. Maydole (1975) generates paradoxes for S_n and S_n* which indicate grounds for ruling out (40) and (41) as implication functions. Both (44) and (45), leading to (42) and (43) have their intuitive attractions and need further investigation in practical situations. In the examples in the following section it will be found in general that either will suffice and, indeed, for most examples the common inference rule (35) is adequate without further specification.

It is interesting now to consider cases in which the implication itself is not precise. Suppose $c=a\supset b$ but $c\neq 1$, $c=1-\varepsilon$, say. What can we infer about the relationship between a and b. Here again the defects of (40) and (41) become apparent because (40) will not allow this case and (41) will not allow it if $a\leqslant c$. Definitions (42) and (43) however give us freedom to fix c and a without constraint and then infer an inequality for b. From (42) we obtain:

$$b\geqslant a-(1-c) \tag{46}$$

and from (43):

$$b\geqslant a\times c. \tag{47}$$

We can actually derive the stronger result that these inequalities become equalities when $c\neq 1$. This seems rather too strong, however, allowing us to infer the truth value of b from those of a and $(a\supset b)$, and Gaines (1976*b*, *c*) uses the rather weaker result that (46) and (47) may be derived from the general forms (44) and (45), respectively, and the general result that if $x=a\wedge b$, $x\leqslant a$, i.e. the min/max connectives for conjunction/disjunction are not assumed. Alternatively we might assume that the truth value for a second-order concept of implication is known only itself as an inequality, $c\geqslant 1-\varepsilon$, which would again make the inequalities of (46) and (47) proper under all conditions.

Various other operations may be defined, or introduced, in the logic, e.g. if we take the **PC** disjunction of (25) but use the implication of (42) we obtain:

$$d={\sim}a\vee b\rightarrow d=\min(1,\, a+b), \tag{48}$$

an operation which Zadeh calls the "*bounded sum*". If we take the negation of the implication of (42) we obtain:

$$c = \sim(a \supset b) \rightarrow c = 1 - \min(1, 1 - a + b) = \max(0, a - b) \quad (49)$$

which Zadeh calls the "bounded difference".

Whilst "bounded sum" and "bounded difference" are definable within the framework of Łukasiewicz $Ł_1$, Zadeh also makes extensive use of a new unary operation (which I shall write as γ) to represent the hedge "very". This is additional to $Ł_1$ and has the form:

$$b = \gamma a \rightarrow b = a^2. \quad (50)$$

We clearly have:

$$\gamma x \supset x \quad (51)$$

which may be read that the truth value of something being "x" is at least that of its being "very x". However, γ may also be read formally as a unary operation extending $Ł_1$.

In summary a logic may be developed based on fuzzy set theory in which the degree of membership of an element to a set is taken to be the truth value of the statement that that element belongs to the set. The appropriate connectives and inference rules for the logic then depend on the definition of union, intersection, and complementation of fuzzy sets, and the inequality of (35) which is a form of *modus ponens* for **MVLs**. If implication is to be allowed in statements then its valuation must be defined consistent with (35), for example, by (44) or (45).

It is interesting to move outside the arithmetic context for fuzzy logics established so far and consider their axiomatic forms. Many classical logics do *not* have representations as truth-functional many valued logics, and it is easier to see their relationship to fuzzy logics from the relative axiom schemes. It may also be a more intuitively satisfying foundation for the fuzzy logics to develop them non-numerically as patterns of reasoning and then re-introduce "degrees of membership" as a mathematical (or computational) device using Wajsberg's (1931) technique.

Zadeh's standard connectives for fuzzy sets together with the implication of (44) give Łukasiewicz infinite valued logic, $Ł_1$. For comparison purposes, this is best axiomatized by a scheme involving only implication:

$$\frac{a \quad a \supset b}{b} \quad (\textit{modus ponens}) \quad (52)$$

$$\frac{\dagger}{a} \quad \text{(contradiction)} \quad (53)$$

$$\frac{*}{a \supset (b \supset a)} \quad \text{("paradox")} \quad (54)$$

$$\frac{*}{((a \supset b) \supset a) \supset ((b \supset a) \supset a)} \quad \text{(disjunction)} \quad (55)$$

$$\frac{*}{(a \supset b) \supset ((b \supset c) \supset (a \supset c))} \quad \text{(transitivity, 2nd syllogism)} \quad (56)$$

$$\frac{*}{\forall x\, Fx \supset Fa} \quad \text{(instantiation)} \tag{57}$$

$$\frac{a \supset Fx}{a \supset \forall x\, Fx} \quad \text{(generalization)} \tag{58}$$

where a and b are propositions, F is a propositional function, and x is an individual variable that does not occur free in a. In this scheme (Kneale & Kneale, 1962) * denotes that the null proposition may be replaced by that below the line, and † is a false, or contradictory, proposition. I have introduced quantification for convenience at this stage as it fits naturally in the axiom scheme and will be needed later. The remaining logical connectives of quantified $\mathbf{Ł}_1$, $\mathbf{QŁ}_1$, may be introduced by the definitions:

$$\sim a \text{ for } a \supset \dagger \quad \text{(negation)} \tag{59}$$

$$a \vee b \text{ for } (a \supset b) \supset b \quad \text{(disjunction)} \tag{60}$$

$$a \wedge b \text{ for } \sim(\sim a \vee \sim b) \quad \text{(conjunction)} \tag{61}$$

$$a \equiv b \text{ for } (a \supset b) \wedge (b \supset a) \quad \text{(equivalence)} \tag{62}$$

$$\exists x\, a \text{ for } \sim(\forall x) \sim a \quad \text{(existence)} \tag{63}$$

All the definitions and inference rules of this schema are valid in the standard lower predicate calculus, **LPC**, so that $\mathbf{Ł}_1$ may be regarded as a weakened form of **PC** and quantified $\mathbf{Ł}_1$, $\mathbf{QŁ}_1$, as a weakened form of **LPC**. The missing rule for **LPC** is:

$$\frac{*}{((a \supset b) \supset a) \supset a} \tag{64}$$

and addition of this to (52)–(56) gives **PC** or, with (57)–(58), **LPC**. It is probably more familiar as *consequentia mirabilis* with † replacing b:

$$\frac{*}{(\sim a \supset a) \supset a} \quad \textit{(consequentia mirabilis)} \tag{65}$$

which from definition (60) may be written

$$\frac{*}{(\sim a \vee a)} \quad \text{(excluded middle)} \tag{66}$$

which is definitely not part of $\mathbf{Ł}_1$. Note that (55) may be regarded as a weakened form of (64), particularly if written:

$$\frac{*}{((a \supset b) \supset a) \supset (a \vee b)}. \tag{67}$$

Ackermann (1967) notes that one consequence of the lack of (64) is that a common deduction theorem is missing:

$$(a \supset (a \supset b)) \supset (a \supset b) \tag{68}$$

which would allow the deduction of $a \supset b$ from the hypothesis of a to allow us to drop the hypothesis of a and still retain $a \supset b$. This highlights the care needed with $\mathbf{Ł}_1$ and $\mathbf{QŁ}_1$ to avoid importing results of **PC** and **LPC** which are no longer valid. It is the same problem that exists when working within the axiom scheme of other variants such as the intuitionistic propositional calculus (**IPC**) or the Lewis-Langford modal logics.

Whilst the addition of (64) or **LEM** to $Ł_1$ takes us back to **PC**, there are other alternative logics that do *not* contain **LEM** also.† It is interesting to establish the relationship of these to $Ł_1$. If one compares (54)–(56) with the implication fragments of the modal logics **S2** through **S5** (Lemmon, Meredith, Meredith, Prior & Thomas, 1969) then (54) is regarded as a "paradox" of strict implication but (68) is a theorem of the modal logics. This deduction theorem is available in **IPC** also stemming from the axiom:

$$(a\supset(b\supset c))\supset((a\supset b)\supset(a\supset c)) \tag{69}$$

which is the *only* one of the 10 (Łukasiewicz) **IPC** axioms (Rescher, 1969) that is not a theorem of $Ł_1$. Conversely from (55)–(56) we can derive:

$$((a\supset\dagger)\supset\dagger)\supset a \tag{70}$$

i.e.

$$\sim(\sim a)\supset a \tag{71}$$

which is a key non-thesis of **IPC**.

Thus in terms of the stocks of tautologies **S2–S5**, **IPC** and $Ł_1$ each contain *different* fragments of **PC**. The lack of (68) in $Ł_1$ compared with the others illustrates the wider role of implication in a system with graded truth values—that the hypothesis, a, may be at a low enough level of truth to derive $a\supset b$ does not allow one to unconditionally assert $a\supset b$, e.g. $a=0{\cdot}5$, $b=0{\cdot}4$, gives $(a\supset(a\supset b))$ a truth value 1, but $a\supset b$ a truth value 0·9. It is interesting to recall Mostowski's remarks (1966, p. 17) to the effect that only **IPC** so far satisfied Łukasiewicz's program of developing alternative logics that would actually be *used* by working scientists. Developments in recent years in modal logics and their applications, and now in fuzzy logics and their applications, indicate that the use of alternative logics (a key factor is evaluating their utilities!) is now spreading.

So far as I am aware, no axioms characterizing γ or the alternative form of implication, $-\!\!\ni$ (45), have been given‡ although several authors have studied conditional probability-like valuations as measures of implication (e.g. Adams, 1966; Danielsson, 1967; Stalnaker, 1970)—these are interesting open questions. A multivalued interpretation of the quantifiers may be introduced, consistent with (57) and (35), that:

$$a=\forall xb\rightarrow a=\inf(b'), \tag{72}$$

$$a=\exists xb\rightarrow a=\sup(b'), \tag{73}$$

where b′ is any substitution instance of b. Modalities are also readily introduced into the QL_1 schema using a conventional "possible worlds" model (Snyder, 1971). Since much of fuzzy reasoning is concerned with sequential structures, automata (Gaines & Kohout, 1975) and socio-dynamic systems (Wenstøp, 1976; Kohout & Gaines, 1976),

†Note that the modal logics considered here do not contain "**LEM**" as defined here because the implication used in its definition is the *strict* form, i.e. necessary implication.

‡It appears quite feasible to axiomatize γ and $-\!\!\ni$ within the framework of $Ł_1$. For example:

$$\gamma(\sim a\supset b)\supset(\sim\gamma a\supset\gamma b)\equiv(\sim\gamma a\supset\gamma b)\supset\gamma\sim(a\supset b)$$

catches the essence of γ. Whilst:

$$a-\!\!\ni(\sim b\supset c)\equiv\sim(a-\!\!\ni b)\supset(a-\!\!\ni c),$$
$$\gamma(a-\!\!\ni b)\equiv(\gamma a-\!\!\ni\gamma b)$$

do the same for $-\!\!\ni$.

the possible worlds have natural and important interpretations as "reachable" or "potential" states of affairs. In particular, the stability theory of fuzzy linguistic controllers needs the fuzzy tense logic equivalent of Prior's (1967) interpretations of **S4**. However, this is again an area of great practical importance that appears to be undeveloped as yet.

7. Analysis of some paradoxes

Before moving on to the linguistic aspects of fuzzy reasoning, particularly the linguistic correlates of logical terms, it is worthwhile analysing how some of the classical "paradoxes" of logic and set theory may be resolved using logics based on fuzzy set theory. Russell's paradox and that of sorites illustrate different aspects of the use of a multivalued logic as a basis for set theory and reasoning.†

Consider Russell's well-loved barber who performs the (very socially desirable) duty of shaving those people who do not shave themselves, but becomes very puzzled when he tries to work out whether to shave himself or not‡. Let bshav be the (fuzzy) set of people who are shaved by the barber and sshav be the (fuzzy) set of those who shave themselves. That the barber shaves those who do not shave themselves (not shaving implies barber shaves) gives us:

$$\overline{\text{sshav}\mathbf{x}} \leqslant \text{bshav}\mathbf{x} \tag{74}$$

so that

$$1\text{–sshav}\mathbf{x} \leqslant \text{bshav}\mathbf{x}. \tag{75}$$

That those who shave themselves are not shaved by the barber gives us, similarly:

$$\text{sshav}\mathbf{x} \leqslant 1\text{–bshav}\mathbf{x}. \tag{76}$$

Taking inequalities (75) and (76) together we have

$$\text{sshav}\mathbf{x}+\text{bshav}\mathbf{x}=1. \tag{77}$$

However, now consider the barber: his membership to the set of those who shave themselves is clearly the same as that to the set of those shaved by the barber (because that is himself—those who feel this step is dubious can go away and invent a theory of types to forbid it!). Hence, for the barber, **b**:

$$\text{bshav } \mathbf{b}=\text{sshav } \mathbf{b}=1/2. \tag{78}$$

This value 1/2 is a third truth value that is not available in a bivalent logic, but becomes so in an appropriate trivalent logic as noted by Shaw-Kwei (1954), Skolem (1957) and Varela (1975), in the context of classical set theory, and by the late W. L. Hendry (1972) in the context of fuzzy set theory.

The analysis of the barber paradox shows us that the barber is a different *type* of individual from the others, in fact a mid-range case at exactly the *borderline* between the two degrees of membership (0 and 1) previously allowed. It is interesting to note

†The analysis of paradoxes is a fruitful test of techniques for formalizing imprecise reasoning. Martin (1970), Weiss (1973), and Hughes & Brecht (1976), contain interesting discussions of suitable paradoxes.

‡The actual barber on whom the anecdote is based did not cut his throat as is often reported, but lived to a peaceful old age when, becoming bald of face, he lost interest in the problem.

that the logical argument has forced us to *generate* a new truth value, i.e. the possibility of 1/2 has not been introduced through an arbitrary extension to a ternary logic, but forced from the inequalities of implication, (75) and (76). It is the natural way in which new truth values are generated as needed that makes this approach so attractive†.

The solution to the paradoxes of sorites, falakros, etc., is of a rather different nature. Let us consider what it is to be a bald man‡. Most bald men are not completely hairless yet we would still call them bald. Suppose a stranger comes along with only one more hair than one of our bald men. We would clearly call him bald also (although this may be dangerous with strangers). But then another stranger comes along with one more hair than this one, and so on—there is no reasonable way in which our criterion for baldness can depend on the possession of one additional hair and yet, before you can say hello to the ten millionth man, you are calling very shaggy men bald.

Let $\mathbf{hair}_n$ designate a person with n hairs on his head (the boundaries of heads and the enumerability of hairs will be here taken to be precise, not fuzzy) who has a degree of membership, bald $\mathbf{hair}_n$, to the fuzzy set of bald people. The key inference is that a person with only one more hair than a bald man is still bald, so that:

$$\forall\, n, (\mathbf{hair}_{n-1} \text{ is bald}) \supset (\mathbf{hair}_n \text{ is bald}). \tag{79}$$

Thus, using (35), we have:

$$\forall\, n, \text{bald } \mathbf{hair}_n \geqslant \text{bald } \mathbf{hair}_{n-1} \tag{80}$$

which gives us, by induction on n:

$$\forall\, n, \text{bald } \mathbf{hair}_n \geqslant \text{bald } \mathbf{hair}_0. \tag{81}$$

This, remembering that we take the truth value of a statement such as, "a man with n hairs on his head is bald" to be the degree of membership of the man to the (fuzzy) set of bald men, bald $\mathbf{hair}_n$, may be read as: "it is at least as true that a man with an arbitrarily large number of hairs is bald as that a man with no hairs is bald"—certainly a paradox!—whence does it derive?

The weakness is in the transition from (79) to (80) since this is derived from (35) which is valid only if the implication of (79) is itself precise, i.e. absolutely true. When asked to agree to the statement that, "a man with only one more hair than a bald man is still bald", we must surely have our doubts—it is an argument "in the wrong direction", but not by much. We might answer, "Well, I suppose so", and feel that it would be somewhat pedantic to express too strong a doubt. If now told that truth is bivalent, "Be more precise. Is it true or not?", we are forced to make a choice. We cannot say that the statement is not true—it is far more reasonable to suppose it true than false, and, if these are the only possible choices, then we must say it is true.

Thus the demand for precision and bivalency of truth forces us to a choice and hence to a paradox. What we really wish to say is, "Yes, that is true", "Yes, that is quite true", or even "Yes, that is very true". "Truth" is itself a fuzzy concept that we *hedge* in everyday usage. I will discuss this further in the following section, but let us for the moment take it that truth is *not* bivalent and that the implication of (79) has itself a truth value of less than 1. Then the proper inference from (79) is not (80) but one based on (46) or (47)

†I am grateful to Francisco Varela for this insight.

‡It may well have been the same barber, in deciding who needed shaving at all, who came up with this problem also. Debate on this point would be fruitless, or at least only a matter of splitting hairs.

according to which implication function we are using, i.e. if the truth value of implication in (79) is 1–ε:

$$\forall\ n,\ \text{bald}\ \mathbf{hair}_n \geqslant \text{bald}\ \mathbf{hair}_{n-1} - \varepsilon, \tag{82}$$

or

$$\forall\ n,\ \text{bald}\ \mathbf{hair}_n \geqslant \text{bald}\ \mathbf{hair}_{n-1} \times (1-\varepsilon), \tag{83}$$

which by induction on n gives us

$$\forall\ n,\ \text{bald}\ \mathbf{hair}_n \geqslant \text{bald}\ \mathbf{hair}_0 - n \times \varepsilon, \tag{84}$$

or

$$\forall\ n,\ \text{bald}\ \mathbf{hair}_n \geqslant \text{bald}\ \mathbf{hair}_0 \times (1-\varepsilon)^n. \tag{85}$$

Note the interesting forms of inequalities (84) and (85): the left-hand side goes to zero with n, either linearly or exponentially; however, the right-hand side is greater than or equal to this decreasing quantity. We can no longer infer that the truth value of a man with an arbitrarily large number of hairs being bald is at least that of a man with none. However, neither can we infer that the truth value tends to zero, only that the lower bound on it obtained through our long chain of reasoning is zero and hence non-informative. It is the long chain of reasoning that is worthless (one is tempted to say invalid—however it is the false inference of (81) that is invalid). If we are to say anything about the baldness of a man with n hairs on his head it is certainly not through this chain of reasoning.

This analysis of falakros raises many interesting points. It may be contrasted with that of Lake (1974) who finds no satisfactory analysis in terms of fuzzy set theory but is still looking for a definite criterion of baldness—the argument chain does not break at some definite point—rather its information content peters out as it becomes longer. And this, in terms of the discussion of section 1, is just what one hopes an adequate account of reasoning with imprecise concepts will do with sorites, falakros, etc.—defuse them rather than use them as a source of further artificial precisiation. The paper does not suddenly become irrelevant to control engineering—rather its relevance drops to a point where it is no longer robust against the random actions of referees, editors and readers, and it disappears quietly rather than through a dramatic application of new and precise criteria!

The point that the implication of (79) is not itself to be assigned a truth value of 1 is highly significant. I believe that this will be found to be of key importance in all aspects of practical reasoning. Each step we take in the chain of reasoning downgrades our estimate of the truth value of the result. The form of *modus ponens* in fuzzy reasoning is always like (46) or (47) and not the precise step of (35). This raises a methodological point for studies of human reasoning. If we have two premises, a and b, of given truth value then inferring the truth value of c=avb is not just a matter of using a rule such as:

$$c = \max\ (a, b) \tag{86}$$

but rather one of using the *inferences*:

$$a \supset c,\ b \supset c \tag{87}$$

and hence involves a chain of reasoning and a potential downgrading of the truth value of the result. In fuzzy reasoning we "pay" for each application of *modus ponens*. The long chain of argument eventually peters out even if every premise is known to be precisely true—we have a small, but not negligible, mistrust of each step. This gives a very tangible significance to the, otherwise, only aesthetic principle in logic that shorter proofs are to be preferred to longer ones (a principle long accepted, with little justification, by referees and commentators!).

8. Fuzzy reasoning, hedges and truth

The analysis of the paradoxes in the previous section shows how they can be avoided in natural, and constructive ways, using reasoning based on a multivalued logic. This, in itself, may be viewed as illustrating only how the "classical" non-standard approaches to these paradoxes may be regarded as part of fuzzy reasoning. To go beyond this we need to develop further the *linguistic* aspects of membership in fuzzy sets, *hedges*, and Zadeh's theory of *truth*.

First of all we will formally adopt the conventions previously suggested that: expressions of properties in the form "x is y" may be translated into logical statements of the form "**x** is a member of the fuzzy set y"; and hence the degree of membership of **x** to y, y**x** is meaningful and corresponds to the truth value of the original statement, "x is y". This relationship between linguistic forms and fuzzy sets is important because it enables us to go on to consider linguistic extensions such as "x is y *if* u is v", "x is *very* y", " 'x is y' is *true*", etc.

It is customary in exposition of fuzzy reasoning to introduce at this stage some "examples" of fuzzy sets, such as the set of *tall men*, and to do so by prescribing a a degree of membership to this set for a particular man as a function of his height. For illustrative purposes, in that it allows one to draw diagrams of fuzzy sets and their modifications, this procedure is useful. However, it has a danger of putting the ontological cart before the phenomenological horse—the concept, the perception, of "tallness" exists in a more primitive sense than does the measurement of "height"—to derive, even for illustrative purposes, the concept *tall* from measurements of the physical variable *height* can be misleading. We are able to generate and follow arguments involving "tallness" without having any concept of inches, centimetres, or any other metric scales. To introduce the former in terms of the latter reverses the actual process of derivation and, in particular, leads to a false distinction between these concepts, such as "tallness", that have a well-defined, single-parameter, physical metric, and those, such as "beautiful" which do not.

Thus, whilst a "scientific" analysis might conclude that there is a wide and ill-defined range of physical phenomena that combine in an extremely complex fashion to produce the subjective impression of beauty, in everyday reasoning it is as primitive a term as the more simply explicable tallness. We certainly do not distinguish between them in arguments such as:

He likes girls that are tall and beautiful.
Mary is not very tall but very beautiful.
He will probably like Mary.

or

This ladder is too tall to put in the kitchen.
This vase is too beautiful to put in the kitchen.

I emphasize this point at this stage because it is rather easier to illustrate fuzzy reasoning with linguistic variables that do have a simple physical interpretation—it is possible then to validate the results by comparing them with a more scientific argument. However, there is a definite tutorial danger that the reasoning pattern thus exhibited becomes associated with such variables, "tall", "heavy", "wide", etc., and terms such as "beautiful", "angry", "peaceful" are seen as less amenable to formal inference. Such an association would be very wrong—it is the main motivation of studying and formalizing fuzzy reasoning that it allows equal facility of inference with such "non-physical" and "ill-defined" terms as one normally has with those more amenable to precisiation.

Thus the concept of degree of membership to a fuzzy set is primitive. If we are able to go beyond this and define an operational procedure leading to a number that, over a population, is isotonically related to the degree of membership all well and good. We then have a physical measure that corresponds to the concept and which may itself have further properties that make the measuring scale more tractable (Krantz *et al.*, 1971). It is the isotone relationship between measurement and degree of membership that makes for "pretty" diagrams, but it is also this same factor which makes the measurement useful in explicating the concept, not vice versa. This may be seen clearly, for example, in multidimensional scaling techniques (Shephard, Romney & Nerlove, 1972) where the computation is constrained to force data into the lowest dimensions of physical measurement that enable the order relationship between distances between points to be maintained, i.e. the fuzzy concept of "degree of similarity" is the *primitive from which measures are derived.*

In the light of these arguments I shall develop Zadeh's theories of *hedges*, and of *truth*, in rather more formal terms than usual, and exemplify them initially without introducing actual, numerically-defined fuzzy sets. This approach also serves to distinguish those patterns of fuzzy reasoning that are tautologous properties of the definitions, and those which depend on the empirical properties of actual fuzzy sets.

In the same way that **PC** may be seen as an attempt to formalize colloquial linguistic usage of terms such as "and" and "or", and various modal logics as attempts to formalize "possible" and "necessary" or "permissible" and "obligatory", so can much of current work on fuzzy logic be seen as formalizing linguistic usage of certain *hedges* applied to imprecise concepts, such as "very", "more or less", "slightly", etc.† (Lakoff, 1975; Zadeh, 1972*c*). Before launching into technical detail it is worth re-issuing the usual disclaimer that must be made when attempting to parallel the colloquial use of language in a logical formalism:

> "it is useful to attempt to concretize the meaning of a hedge such as very even if the postulated meaning does not have universal validity and is merely a fixed approximation to a variety of shades of meaning which very can assume in different contexts" (Zadeh, 1972*c*).

Note, in passing, that this disclaimer is itself a fuzzy metalinguistic statement. When it comes to precisiation there is a sense in which fuzzy logic is the *only* one which can provide its own metalanguage!

Zadeh (1972*c*) suggests that the hedge *very* is interpreted as the operator, γ, introduced in section 6. Thus if a is the statement:

$$\text{"The vase is beautiful"} \qquad (\text{vase} \in \text{beautiful objects}):$$

†For a variety of linguistic approaches to hedges see the series edited by Kimball, Cole & Morgan (1972–75).

then γ a is the statement:

"The vase is very beautiful" (vase $\in$ very beautiful objects).

This new unary operator can combine with the normal "and", "or", "not", operations, to produce very complex predicates. The statement, b:

"The vase is very beautiful but not very very beautiful"

is

$$\mathrm{b}=\gamma\mathrm{a} \wedge {}^{\sim}\gamma\gamma\mathrm{a} \tag{88}$$

so that, from (31), (33), and (50):

$$b=\min(a^2,1-a^4). \tag{89}$$

Clearly, particularly when compounded with further statements, such as "the vase is green", to give "the vase is not very green, but very very beautiful", very complex numerical relationships between truth values can develop. Zadeh points out the way in which language develops to give short forms of the commonly used compound hedges. "Very not beautiful" is ungrammatical and we replace "not beautiful" with "ugly" allowing "very ugly". A compound such as "the vase is beautiful but not very beautiful" may be condensed to "the vase is *slightly* beautiful", i.e. if we represent "slightly" by the unary operator, σ, then:

$$\sigma\mathrm{a}=\mathrm{a} \wedge {}^{\sim}\gamma\mathrm{a}. \tag{90}$$

To deal with other hedges, Zadeh introduces a range of other operators that may be defined in terms of γ. Since γ itself decreases the degree of membership most for those elements that already have a low degree of membership he calls it "*concentration*". "*Dilation*", δ, has the opposite effect and is used to express the hedge "more or less". It can be defined implicitly as:

$$\mathrm{a}=\delta\mathrm{b} \Leftrightarrow \mathrm{b}=\gamma\mathrm{a} \tag{91}$$

we clearly have:

$$a=b^{0\cdot5}. \tag{92}$$

"*Intensification*", ι, pushes values above 0·5 towards 1 and values below 0·5 towards 0, i.e. it decreases the "fuzziness" of the set. It may be defined as:

$$\mathrm{a}=\iota\mathrm{b} \Leftrightarrow \mathrm{a}=(({}^{\sim}\gamma\mathrm{b} \supset \gamma\mathrm{b}) \wedge \mathrm{b}) \vee (\gamma^{\sim}\mathrm{b} \supset {}^{\sim}\gamma^{\sim}\mathrm{b}) \tag{93}$$

so that:

$$a=\begin{cases} 2\times b^2 \text{ if } b\leqslant 0\cdot5, \\ 1-2\times(1-b)^2 \text{ if } b\geqslant 0\cdot5. \end{cases} \tag{94}$$

Thus "concentration" and "dilation" are essentially compound hedges defined in terms of the connectives of Ł_1 with the added operator, γ. Zadeh also finds it useful to introduce a "normalization" operator, ν, that may be used to ensure that a fuzzy set is normalized, i.e. attains a maximum degree of membership of 1. In terms of the connective, $\rightarrow\!\!\ni$, of (45) we may define:

$$\mathrm{a}=\nu\mathrm{b} \Leftrightarrow \mathrm{a}=(\exists\mathrm{x}\ \mathrm{b}) \rightarrow\!\!\ni \mathrm{b} \tag{93}$$

so that:

$$a = b/(\sup b') \tag{94}$$

where b′ is any substitution instance of b.

The hedge "sort of" can be defined in terms of ν and δ as:

$$a = \nu(\gamma\gamma b \wedge \delta b) \tag{95}$$

where a might be "the dog is sort of fierce" and b "the dog is fierce". Lakoff & Zadeh define many other hedges in this way, "pretty", "rather", "highly", etc. An important feature of their approach is that the hedges serve to derive new fuzzy sets in a standard fashion for those originally defined. If we know what is meant by "beautiful" then we also know what is meant by "very beautiful", "slightly beautiful", "not beautiful", and so on. The hedges clearly represent a way of deriving new fuzzy predicates in a standard fashion, independent of the structure of the particular fuzzy sets to which they are applied.

In general a chain of reasoning will result in some fuzzy predicate that will often be expressible linguistically in terms of standard hedges. For example, if we can finally assert, $\sim\gamma\gamma b \wedge \delta b$, then if b is "my hair is wet", we can state "my hair is sort of wet". However, there are far more combinations of connectives possible than there are standard linguistic hedges in our vocabulary, and Zadeh (1975) introduces the notion of "*linguistic approximation*", in which the function expressed by a linguistic hedge does not precisely correspond to that resulting from some reasoning but is a good approximation to it. Linguistic approximation may be expressed axiomatically as a binary predicate, λ, between two fuzzy predicates, a and b, which are themselves functions of the free variable x:

$$\lambda(a,b) = \forall x(a \equiv b) \tag{96}$$

so that the degrees of membership of a and b to the fuzzy set of linguistic approximations are 1 minus the maximum difference between them over the domain of the fuzzy set on which they act.

The concept of *linguistic approximation* provides a clear example of the phenomenon mentioned earlier whereby fuzzy logic acts as its own metalanguage. The statement that "*very but not very very* is a linguistic approximation to *rather*" is itself imprecise and requires formalization in terms of fuzzy reasoning. The expression in (96) shows that this formalization is available within the framework already developed, and enables us immediately to formulate metalinguistic arguments about fuzzy reasoning such as: "*rather* is not a very good linguistic approximation to *very but not very very*". This metalinguistic completeness is one of the most satisfying features of the theory of fuzzy reasoning development by Zadeh. It is a *de facto* rejoinder to those who argue that bivalent logics must be foundational because we use them to argue about multivalued logics—in reality this is just false—"precisiation" itself is most naturally discussed and analysed in a fuzzy metalanguage.

This is a natural point at which to discuss Zadeh's theory of truth (Bellman & Zadeh, 1976) which takes *truth* itself to be a hedgeable linguistic concept, *not* an absolute, bivalent entity. I have already suggested in connection with the falakros type of paradox that our answer to the question, "Is a man with one more hair than a bald man still bald", might be, "Yes, that is very true". Zadeh treats such a statement as expressing a relationship between the truth value of the first statement and the (fuzzy) set of truth statements. Thus the truth value of the second statement is the degree of membership of the truth value of the first statement to the set of true statements.

This concept is very clear and elegant—it enables us to analyse immediately statements of the form: " 'a' is true", [" 'a' is true" is more or less true], "it is not true that 'rather' is a linguistic approximation to 'very more or less' ", and so on. By placing appropriate constraints upon the function between the value of a statement, a, and its degree of membership to the set of true statements, $\tau(a)$, we can infer some powerful metalinguistic results. For example, it is reasonable to constrain $\tau(a)$ to be such that:

$$\tau(a) \leqslant a \tag{97}$$

in effect, that the truth value does not increase when we say "is true". We then have:

$$\text{“a is true”} \supset \text{a} \tag{98}$$

which is reminiscent of Tarski's convention T (Tarski, 1956). We also have, however:

$$\text{“‘a is b’ is very true”} \supset \text{“a is very b”} \tag{99}$$

because:

$$(\tau(\mathbf{ba}))^2 \leqslant (\mathbf{ba})^2 \tag{100}$$

which is a new type of result in which the hedge of a statement about truth *transfers* to become a hedge of the primary statement, i.e. a hedge in the metalanguage induces one in the object language.

Zadeh gives numerical examples using functions for $\tau(a)$ in terms of a in which the inequality of (97) is proper. We do not then have the inverse inference that: a⊃ "a" is true, and it seems reasonable that this should be lacking. However, it is interesting also to consider $\tau(a)$ to be identical to a:

$$\tau(a) = a \tag{101}$$

in which case:

$$\text{a} \equiv \text{“a” is true.} \tag{102}$$

This particular function has the advantage that one can do without the numeric values in many cases and deal with arguments involving statements about truth in linguistic terms. This itself is significant because the old problem of the meaning of intermediate truth values still persists. " 'Jack is tall' is 0·9 true" raises complications that "'Jack is tall' is very true" does not.

As in any analysis of reasoning in colloquial language there are bound to be counter-examples to the straightforward analysis of hedges and truth so far proposed—both Lakoff and Zadeh give examples where the analysis of this section breaks down. However, there is also a very wide variety of examples of reasoning with imprecise concepts that can now be formalized and understood using their models of hedges and truth. In particular such imprecise reasoning may now be automated in computer programs dealing with the everyday concepts of natural language and decision-making. Logicians who object to inferences of the form: " 'Socrates is healthy' is very true—so 'Socrates is very healthy' ", had better demolish the basis for them very rapidly—tomorrow they will be faced by computer terminals that use such patterns of reasoning in deciding what articles they should read; whether their credit is sound; and whether their divorce requests should be allowed!

I will conclude this section with two examples of fuzzy reasoning in action, the first dependent upon, and the second independent of, actual numeric truth values. Consider the well-known children's poem†:

Fuzzy Wuzzy was a bear
Fuzzy Wuzzy had no hair
Fuzzy Wuzzy wasn't very fuzzy, was he?

Let Fuzzy Wuzzy be represented as **fw**. The intention of the first line is not clear but it probably introduces the concept of a degree of membership, i.e.

$$\text{bear } \mathbf{fw} \approx 1. \tag{103}$$

The second line is clearly a contraction of some longer statement (we say someone has *no* sense, or *no* taste, meaning they have very little) and is probably best precisiated as if it were **fw** *had very little hair* (which does not scan‡) or *was very bald* (which does not rhyme), i.e.

$$\text{very-bald } \mathbf{fw} = 1 - \varepsilon \tag{104}$$

so that:

$$(\text{bald } \mathbf{fw})^2 = 1 - \varepsilon. \tag{105}$$

The final line questions the truth value of very-fuzzy **fw**. It is clear in this context that *fuzzy* is to be taken as *hairy* and hence as *very not bald.* Hence, we have:

$$\begin{aligned} \text{very-fuzzy } \mathbf{fw} &= (\text{fuzzy } \mathbf{fw})^2, \\ &= (1\text{-bald } \mathbf{fw})^4 \leqslant (\varepsilon/2)^4. \end{aligned} \tag{106}$$

Thus finally:

$$\begin{aligned} \text{not-very-fuzzy } \mathbf{fw} &\geqslant 1 - (\varepsilon/2)^4, \\ &\geqslant 1 - \varepsilon = \text{very-bald } \mathbf{fw}, \end{aligned} \tag{107}$$

so that the inference that Fuzzy Wuzzy was not very fuzzy from his lack of hair is valid with a substantial margin, indeed *very valid.* We would be entitled to answer, "Yes, that is most certainly correct".

Note that argument is independent of ε. I could have presented it in terms of order relationships only. The original assertion that Fuzzy Wuzzy has no hair does *not* have to be asserted with truth value 1. In essence our reply is, "Whatever degree of certainty you have in the first assertion will most certainly be more than guaranteed in the last". Thus the argument is in essence a tautology of fuzzy logic reflecting the meanings of the hedges used and not dependent on empirical tests of bald and hairy bears. Most nursery rhymes have a moral and it seems reasonable to suggest that this one introduces an important tautology of fuzzy reasoning to a child at an early age!

As a second example let us examine the concept of height. In the same way that having "no hair" is used conversationally as a term meaning "very little hair", so an apparently precise description of height such as "six feet tall" is used conversationally to mean a

†Whilst everyone seems to know it, the origins of this masterpiece are obscure (see Louis Untermeyer, Ed., *Golden Treasury of Poetry*, Golden Press, 1959).

‡I am grateful to Susan Haack for noting this.

fairly tightly constrained degree of membership peaking at six feet. If we wish to express more precision we would say, "exactly six feet", and less would be expressed by "about six feet" with still less by "roughly six feet". Figures 3(a)–(d) illustrate the form of plots of degree of membership to each of these terms of objects of various heights. Note that we can interpret these curves in terms of truth values by saying that if John is 5 ft 9 in tall then "John is six feet tall" has a truth value of 0·7, whereas "John is exactly six feet tall" has a truth value of 0·1, and "John is roughly six feet tall" has a truth value of 0·95.

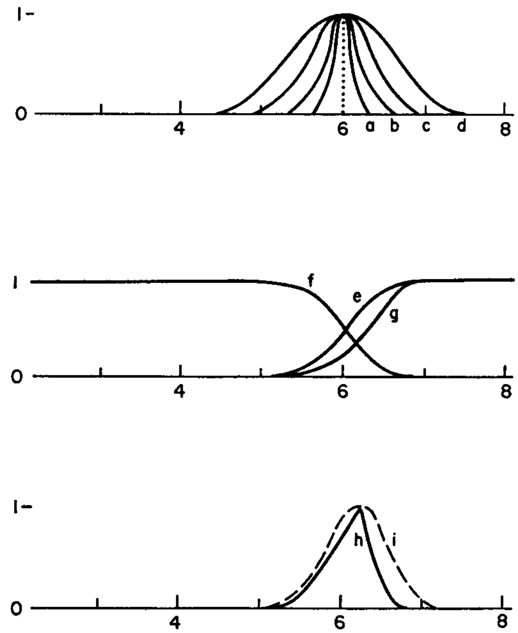

FIG. 3. Variation of membership to various fuzzy sets relating to "tall" against height (in feet). a, "Exactly six feet tall"; b, "Six feet tall"; c, "About six feet tall"; d, "Roughly six feet tall"; e, "Tall"; f, "Not tall"; g, "Very tall"; h, "Tall but not very tall"; i, "About six feet three".

Thus one feature of our normal linguistic expressions of measurement is that they carry with them a connotation of precision as well as one of expected value. They are expressions in *two dimensions*, the *central tendency* and the *degree of precision*, rather than just one. Similar considerations apply to statements about height that do not mention measurements. Figure 3(e) shows the type of variation of degree of membership with height for the fuzzy set specified by "is tall". The exact form of this graph is not likely to be well specified. However, once one is given then the rules of fuzzy logic expressed so far specify how it will change under modification by logical operators and hedges. For example, Figs 3(f) and 3(g) show the graphs for "is not tall" and "is very tall" obtained by the $1-a$ and a_2 rules respectively. Now consider a statement such as "John is tall but not very tall". The normalized combination of logical operators and hedges produces a degree of membership graph shown in Fig. 3(h). This is similar to the dotted graph, Fig. 3(i), for "about six feet three" being a linguistic approximation to it with:

$$\lambda \text{ (about six feet three tall but not very tall)} = 0{\cdot}2. \qquad (108)$$

Thus we can infer that: "John is tall but not very tall" means he is about six feet three tall. This is a further example of fuzzy reasoning, but one that is now crucially dependent on the forms of degree of membership plots.

It would clearly be unreasonable to suppose that terms such as "tall" could be analysed entirely independent of their context—"tall" amongst a tribe of pygmies would not be the same as in a football team! However, in this example, the use of the term would at least be consistent with "tall" for the human race, and it is possible that the membership function within the tribe might be derived from that over the race by taking the normalization function, v of (93), over the tribe. This derivability of membership function *relative to subsets* is an important feature of the order relation of degree of membership, e.g. "young" people in a geriatric population will be "old" in normal terms, but "young" relative to the population and this will appear when their degree of membership is normalized relative to the population.

Another effect one has to take into account is that statements made in conversation do not stand alone but are intended to affect the recipient.† Unless one takes this into account, and hypothesizes that statements are normally made to *convey the maximum information to the recipient*, anomalies can arise. For example, I say "John is tall"—when he appears you look at him and complain, "You misinformed me. John is nearly seven feet. He is not just tall but *very tall*". The principle here seems to be that one chooses the most reasonable description. However, "most reasonable" is *not* the same as "most true"—"John is tall" has a truth value that is substantially higher than "John is very tall". Perhaps the anomaly can be resolved by noting that "John is tall" is very true, but "John is very tall" is just true—we make fuzzy statements at a certain conventional level of truth. It is just as misleading to make them exceptionally true as exceptionally false!

It is clear that a principle of maximum truth is, in any case, inadequate to select amongst statements—we need *relevance* also. The connectives of $\mathbf{QL}_1$ are *not* those of a relevance logic in their current form, e.g. $a \supset b$ is precisely true if and only if $b \geqslant a$, regardless of the relevance of a to b. We have not noticed this defect so far because, as usual, we avoid fallacies of relevance in the way we use the logic. It would be very worthwhile to apply the program developed by Anderson & Belnap (1975) to $\mathbf{QL}_1$ to ensure formally that fallacies of relevance and modality are avoided, particularly in computational forms of the logic.

Thus, there is much to be done before the formal theory of fuzzy reasoning is adequate and complete. However, the concepts developed and examples given do seem to provide a substantial step forward in understanding, and more importantly, emulating human reasoning with imprecise concepts. One important conclusion intended to be drawn from these last two sections is that Zadeh's analysis of fuzzy reasoning, hedges and truth, including such notions as "linguistic approximation", is completely formalizable within a logical framework that is non-standard, but related to classical logics and rigorously expressible. This framework also provides the formal *fuzzy metalanguage* necessary to any discussion of precisiation and vital to the formulation of Zadeh's general arguments about linguistic hedges and truth.

9. Logical, psychological and other models for truth values

One of the key methodological questions posed to proponents of fuzzy reasoning is "where do the numbers come from?" One appears to be replacing fairly simple natural

†See, for example, Grice's theory of meaning (Mackay, 1972).

language inferences with complex operations on fuzzy sets that are themselves defined in terms of fairly complex numeric relationships between degrees of membership and physical properties. Clearly, from the previous sections, one possible answer is that "we can do without the numbers". Fuzzy set theory and logic may be treated axiomatically as a set of inference rules that may be applied to logical expressions in extended $\mathbf{QL}_1$. The fact that connectives in $\mathbf{QL}_1$ can be characterized by operations on numeric "truth values", by infinite valued matrices, is then of secondary interest. It enables fuzzy reasoning to be carried out using numeric computation rather than symbolic processing, an important practical advantage but without foundational significance. This lack of fundamental reliance on numerical operations and intermediate truth values is itself important and I have attempted to illustrate and clarify it in previous sections.

However, from a practical point of view the underlying numeric matrices are an important feature of fuzzy logic, and this section surveys some aspects of the generation of these numbers logically, psychologically and through intuitively meaningful models.

First let us derive the numbers from the axioms. Consider a series of propositions that I shall call *primitive paradoxes*,† p_i for i=0, 1 . . ., defined recursively by:

$$(\sim P_{i+1} \supset P_{i+1}) = P_i,\ i > 0 \tag{109}$$

with

$$p_0 = \dagger \supset \dagger,\ p_1 = \sim P_0 \equiv P_0. \tag{110}$$

In terms of truth values it can be seen that:

$$P_i = (0{\cdot}5)^i \tag{111}$$

so that we have generated an (infinite) sequence of propositional constants with truth values being the binary fractional powers. Now the truth value of any arbitrary proposition may be expressed as a fractional binary expression, e.g. 0·75 decimal=0·11 binary, that may, or may not terminate. Consider expressions of the form:

$$p = (\sim P_i \supset (\sim P_j \supset (\sim P_j \supset (\sim P_k \supset (\ .\ .\ .\ .\ ))))) \tag{112}$$

where $i < j < k < \ldots.$ The truth value of this expression is:

$$p = p_i + p_j + p_k \ldots$$

Hence, if the proposition p_i occurs in the definition of p (112) if and only if there is a 1 in the ith place of the fractional binary expansion of the truth value of a proposition a, then we have: $p = a$. Thus, we can express the truth value of any proposition not as a number but as expression in primitive propositional constants that are themselves directly introduced from our axiom schema for $\mathbf{QL}_1$.

The argument above is readily turned into a formal derivation of a binary numbering system for arbitrary propositional constants that serve to *introduce* the numbers into the system. That is we can prove that if two sequences of the form of (112), p and p′, are such that they commence with the same sequence of primitive paradoxes and then first differ by one being absent from p, we have $p \supset p'$. This enables us to insert arbitrarily long sequences of propositions of the form of (112) between p and a, and a and p′, in relations of the form: $p \supset a \supset p'$, and hence to approximate the truth value of a by a

†These are clearly related to Rosser & Turquette's *J-functions* for axiomatizing the finite-valued versions of Łukasiewicz logic.

Dedekind section of standard sequences of arbitrarily high accuracy. Thus we are able to both do without the numbers in fuzzy reasoning and also to introduce them if required, not on an arbitrary basis, but directly from the axiom schema of (52)–(58).

Rather different approaches to the numerical truth values have been taken by Giles (1976) and Gaines (1976*b*, *c*) in their models of fuzzy logic in terms of *dialogues* and *population responses*, respectively. Giles models $\mathbf{Q}\mathbf{Ł}_1$ as a *game* between two players who engage in a *dialogue* to establish a *pay-off* for a compound proposition that may be used to define its truth value. The connectives are defined by rules such as the following.

(a) He who asserts a ∨ b undertakes to assert either a or b at his own choice.
(b) He who asserts a ∧ b undertakes to assert either a or b at his opponent's choice.
(c) He who asserts a ⊃ b undertakes to assert b if his opponent will assert a.
(d) He who asserts ∃x a undertakes to assert some instance of a at his own choice.
(e) He who asserts ∀x a undertakes to assert some instance of a at his opponent's choice.
(f) He who asserts † promises to pay his opponent £1.

This model can be seen as a set of rules for generating sequents expressed in game theoretic form. The dialogue representation is particularly attractive in some situations, however, and Giles (1975) uses it very effectively to model some of the observational problems of particle physics. In particular he is able to give integrated model-theoretic interpretations of probabilistic and fuzzy decision-making.

This integrated derivation of probabilistic and fuzzy logics is also an important feature of Gaines' (1976*b*, *c*) model in terms of the *responses of a population*. He considers a population, each member of which can respond to certain questions with a binary, yes or no, reply. The forms of question involve evaluating some proposition, a, e.g. "is this proposition, a, true or false, reasonable or unreasonable, believed or not believed?", etc. The value of *a* is defined to be the proportion of the popultion answering "yes". A conjunction of propositions, a ∧ b, is given a truth value in terms of the proportion answering "yes" to both a and b, whilst that for a ∨ b is the proportion answering "yes" for either.

Values for implication are defined as in (44) or (45) and values for equivalence and negation are defined in terms of these. The model is in essence a simple topological one, and its real interest is in terms of interpretation rather than theory as follows.

(a) If we interpret the population as one of physical events and the "questions" as experiments then we have a model of frequentist probability.
(b) If we interpret the population as one of people expressing opinions then the model is a socio-linguistic one of the use of terms by the population—a very reasonable interpretation in terms of "fuzzy reasoning".
(c) If we interpret the population as a population of individual decision-making elements, e.g. "neurons", then the model is one of "subjective probability" or "belief".

Text analysis studies such as those of Rieger (1976) of fuzzy semantic relations in a "population" of documents, fall naturally into this form of interpretation.

The population model so far described gives a non-truth-functional multivalued logic. Gaines (1976*b*, *c*) shows that requiring it to be *strongly truth functional*, so that the value of a compound proposition can be expressed in terms of those of its components regardless of their structure, is sufficient to constrain the logic to be $Ł_1$. He also shows that, alternatively, the law of the excluded middle gives a non-truth-functional logic that is precisely *probability logic* (Rescher, 1969) and goes on to generate a variety of logics by making this truth-functional in different ways. For example, addition of the constraints that, for two atomic propositions, a and b:

(i) $c=a \wedge b \rightarrow c=a \times b$, gives a logic of *statistical independence*;

(ii) $c=a \wedge b \rightarrow c=0$, gives a logic of *mutual exclusion*;

(iii) $c=a \wedge b \rightarrow c=\min(a,b)$, gives a logic of *mutual dependence*.

Gaines suggests that in any particular population different propositions will be found to be related in different ways, e.g. by a fuzzy logic or by statistical independence. If this relationship persists in all relevant variants of the population (possible worlds) then it is a structural constraint and can be expressed in a suitable modal logic. These are the interesting objects to study—the modal logics of fuzziness, independence, exclusion, etc.

One obvious question in terms of the population model is how the constraints leading to a fuzzy logic might arise. Those leading to statistical independence seem somehow more natural and usual. Here there is some interesting psychological evidence relating to interpretation (b) in terms of a population of people. $Ł_1$ would be obtained if members of the population each evaluated the evidence relating to a question in the same way but applied differing thresholds of acceptance when deciding whether to answer "yes". The member with the lowest threshold would then always answer "yes" if any other member did, and so on up the scale of thresholds. This gives a relationship of implication between propositions that generates the min/max connectives of $Ł_1$. This model, although unusual, has its intuitive attractions, e.g. Reason (1969) has shown that the threshold applied by human beings in coming to a binary decision on an essentially analog variable seems to be associated with personality factors and a trait of the individual. If so, human populations would tend to show more a fuzzy, than a probabilistic logic in their decision-making.

There have now also been a number of studies of human decision-making to determine whether "degree of membership" graphs may be obtained consistently from people; what their form is (MacVicar-Whelan, 1974; Kochen & Badre, 1974; Kochen, 1975; Dreyfuss, Kochen, Robinson & Badre, 1975); what the effects of hedges is (MacVicar-Whelan, 1974); and what the effect of logical connectives is (Rödder, 1975). The results of these early studies are encouraging, indicating that the concept of a "degree of membership" is one that can be understood by subjects in a natural way and measured empirically, but they are not yet extensive enough to give definitive answers to questions about the "psycholinguistic reality" of fuzzy sets.

These experiments are reminiscent of those in the psychological literature of: *semantic differentials* (Osgood, Suci & Tannenbaum, 1957); the use of *everyday quantitative expressions* (Sheppard, 1954); subjective decision-making and the elicitation of *subjective probabilities* (Savage, 1971; Shuford & Brown, 1975); *inter-sensory scaling* (Stevens, 1961); and other techniques for obtaining numeric estimates of subjective qualities from human beings. Stevens' (1961) results are particularly interesting because he had subjects give estimates of the intensity of a wide variety of different sensations by the single output of exerting force on a handgrip. The consistency and coherency of his results for a wide range of sensations: shock, warmth, weight, vibration, noise, light, etc., demonstrate that people have no problem transferring qualitative appreciation from one dimension to another, and hence makes it plausible that there may be some high level "logic" of imprecise estimation. Certainly there is much worthwhile work to be done in bringing together these various psychological studies within the framework of fuzzy reasoning. It might well be, for example, that results, such as those of Edwards, Phillips, Hayes & Goodman (1968) that show human subjects as remarkably wasteful of information in a Bayesian decision-making situation, become explicable if people use fuzzy, rather than probabilistic, decision-making logics.

There are also links from fuzzy reasoning to linguistic studies. Wilks' (1975) *preference grammars* and *preference semantics* depend on an order relationship of preference between different possible structures and interpretations, a relationship that arises naturally in fuzzy semantics. His approach is extremely powerful in resolving ambiguous, vague and unlikely constructions, for example, in metaphor and poetry. Thorne's (1969) parser depends on the distinction between a comparatively small and known, set of *closed-class* words, and a potentially infinite and unknown set of *open-class* words. The parts of speech of (unrecognizable) open-class words in a sentence are allocated with reference to the closed-class key-words that are recognized in that sentence. Fuzzy *hedges* fall into the closed class and enable one to recognize that the hedged word is an imprecise predicate. Thereafter one can manipulate it in a reasonable fashion even though one cannot understand it. For example, an ELIZA-type program (Weizenbaum, 1967), could recognize the key word "very" and have reconstruction rules that allowed:

Person: *I am very unhappy.*
Computer: *Is it true that you are unhappy?*
Person: *I dream I am very sad.*
Computer: *Do you dream you are more or less sad?*

which enriches the conversation somewhat!

Thus the formal approach to fuzzy reasoning adopted in previous sections may be given a variety of less formal semantic models and links to human psychology and linguistics. In particular, the numerical values that belong to the multivalued logics underlying fuzzy reasoning may be derived from a number of different sources, ranging from the purely logical to the completely psychological, with a variety of interesting models in between. There is a *plurality* of derivations and which one is appropriate clearly depends on the semantics of the intended application.

10. Summary and conclusions

I am acutely conscious in reading back through this material what a superficial presentation of the subject area it actually gives. What I have been able to do is to give some impression of how developments in fuzzy reasoning might be logically underpinned, not what *are* the foundations, but what *might be* the foundations. This is largely due to the lack of developments in logics such as $\mathbf{QL}_1$ comparable to those in the classical predicate calculus.† David Sanford (1975) quotes C. S. Peirce (1931):

"Logicians have been at fault in giving Vagueness the go-by"

and Saul Kripke (1970):

"Logicians have not developed a logic of vagueness."

†Maydole (1972) and Scott (1975) are the best modern discussions of $\mathbf{QL}_1$.

Future historians will some day undoubtedly puzzle over this lack of interest, and certainly will need to explain the explosion exhibited in Fig. 1 triggered off by Zadeh's seminal paper of 1965.

Why the lack of interest in vagueness? Logical positivism is blamed for so much nowadays that chalking up another black mark will scarcely be noticed. Part of the drive of that movement was undoubtedly to greater precision (Popper, 1976). However, there is nothing in the formal presentation of the positivist approach to scientific theories that precludes the standard predicate calculus in the *Received View* (Suppe, 1974, p. 16) being replaced with a weaker logic such as $\mathbf{Q\L}_1$. It was presumably the influence of Whitehead & Russell's *Principia* (1910–1913) that made the choice of logic obvious. Hopefully now the need to develop weaker underlying logics for set theory, coupled with the need to avoid artefacts of precisiation, coupled with the need to model natural language reasoning, is focusing the attention of philosophers and logicians of mathematics, science and language in the same direction and will lead to rapid progress.

This still does not explain the growth of interest in fuzzy reasoning amongst engineers and social scientists. Suppe (1974, p. 19) has the wry footnote:

> "It seems to be characteristic, but unfortunate, of science to continue holding philosophical positions long after they are discredited"

and that may be a sufficient explanation. Certainly precisiation has proceeded apace, and still proceeds, in systems engineering, linguistics, psychology and sociology, where the development of fuzzy reasoning has been most welcomed. These are the new sciences, *post* von Neumann and the computer, *post* Norbert Wiener and cybernetics, but they have tended to model themselves on the established, and highly successful, physical sciences, just at the point when these sciences themselves were entering an era of self-doubt and review of foundations (Körner, 1957).

What is particularly interesting in applied studies of fuzzy reasoning, such as those as diverse as Mamdani & Assilian's (1975) of process control and Wenstøp's (1976) of social dynamics, is the emphasis on reasoning in natural language. This is taken to be, not just a remote model, but instead an almost directly manipulatable representation of data. One is taken back to debate about the role of formal logic in relation to language about which John Stuart Mill remarks in his *System of Logic* (1843):

> "Since reasoning, or inference, the principal subject of logic, is an operation which usually takes place by means of words, and in complicated cases can take place in no other way: those who have not a thorough insight into both the signification and purpose of words, will be under chances, amounting almost to certainty, of reasoning or inferring incorrectly."

One may speculate that the growth of formal logic, despite its poverty in relationship to natural language, was *not* due to any deep philosophical developments but rather to the lack of computers. As an analogy, we have developed and used linear systems theory not because it works but because it is the *most powerful theory that we could utilize before computers were available*. Similar considerations apply to the classical propositional calculus—its value is largely computational due to its characteristic bivalent matrices. $\mathbf{Q\L}_1$ and similar logics without such a simple representation are too difficult to formalize by hand—see Prior (1967, Chapter 2) to get some feeling for the type of effort involved. Snyder (1971, p. 12) remarks on the dramatic change that the computer has made to such studies:

> "The high adventure of seeking clever strategies for deductive proofs . . . lost to us in the present set of formal systems. Instead, the adventure . . . lies in the development of a variety of systems of logic for a variety of tasks."

Given a computer we no longer need the simplicity, uniformity and analyticity, that is vital to paper and pencil implementation of formal reasoning. We can in fact take natural language at its face value and implement its diverse patterns as they are. This seems far closer to Leibniz's goal (Burks, 1975, p. 299):

> "if we could find characters or signs appropriate for expressing all our thoughts as definitely and as exactly as arithmetic expresses numbers or geometric analysis expresses lines, we could in all subjects in so far as they are amenable to reasoning accomplish what is done in arithmetic and geometry".

rather than his own *universal language* of numbers. Natural language is not precise in itself but it does "exactly express our thoughts"!

A further attraction of taking the reasoning patterns of natural language as our model is that the (strange) distinction between deductive and inductive logics disappears. A variety of qualitative differences between patterns of reasoning may still be distinguished encompassing this one distinction, but it no longer has its unique prominence. Again, for the engineer and scientist, the stark presentation of deductive reasoning as vacuous but valid, and inductive reasoning as fertile but unjustifiable, is disconcerting to say the least. Pragmatically one turns to the reasoning patterns of natural language and says "we may never know how they work, but they *do* work, and we will use them".

Clearly, in practice as shown in this paper, formal techniques can, to a large extent, catch up with the pragmatic success of applications of fuzzy reasoning and give a rationalization for it in the classical style. Formalisms have virtues of clarity, communicability and universality, that are valuable provided they do not constrain us unduly. Having used quotations so freely in this paper, I feel inclined to give the last words to an aphorism by a man who has had more experience than any other philosopher (he lived for some 2000 years!):

> "The difference between science and the fuzzy subjects is that science requires reasoning, whilst those other subjects merely require scholarship" (Lazarus Long, quoted in Heinlein, 1974).

Hopefully the direction of the work described in these notes indicates that the scholarship of multivalued logic has a part to play both in the science of reasoning about (rather fuzzy) human linguistic behaviour and in the foundations of science itself.

I have Joe Goguen and Abe Mamdani to thank for wakening my interest in this area by their stimulating studies and conversations, Ladislav Kohout has for many years been a close collaborator in foundational studies of system theory and I am grateful for his help and advice. Animated discussions with Susan Haack at MVL '76 influenced the presentation of this material. Finally, I am grateful to Lotfi Zadeh for conversations at Berkeley in May 1975 that triggered off these studies and for his continued interest and encouragement.

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