loop. Of all 42 possible patterns studied, 26 of them form no output error and do not feed back, which means the threshold is below three. Eleven patterns, each containing at least two input errors, feed back only once, which means that the feedback loop does not generate new output errors. Four patterns, each containing at least two input errors, feed back twice, which may generate one output error by the second feedback. Only one pattern with three 1's in the syndrome register caused by at least three input errors has gone through the feedback loop three times, which may generate up to two output errors depending on the input error pattern. The empirical tests have thus proved that the feedback loop of the new random-error correction code does not propagate errors.

REFERENCES

[1] W. W. Peterson and E. J. Weldon, Jr., *Error Correcting Codes*. Cambridge, Mass.: M.I.T. Press, 1972.
[2] D. W. Hagelbarger, "Continuous digital error correction systems," U.S. Patent 3 227 333, Jan. 4, 1966.
[3] J. L. Massey, *Threshold Decoding*. Cambridge, Mass.: M.I.T. Press, 1963.
[4] A. J. Viterbi, "Convolutional codes and their performance in communication systems," *IEEE Trans. Commun. Technol.* (Part II), vol. COM-19, pp. 751–772, Oct. 1971.
[5] S. Lin, *An Introduction to Error-Correcting Codes*. Englewood Cliffs, NJ: Prentice-Hall, 1970.
[6] J. L. Massey and R. W. Liu, "Application of Lyapunov's direct method to the error-propagation effect in convolutional codes," *IEEE Trans. Inform. Theory* (Corresp.), vol. IT-10, pp. 248–250, July 1964.

Stability and Admissibility of Adaptive Threshold Logic Convergence

B. R. GAINES AND I. H. WITTEN

***Abstract*—It is shown that an adaptive threshold logic element (ATLE) whose weight vector is perturbed reconverges without error provided its threshold is large compared with the perturbation. This result is generalized to a criterion of convergence, based on *admissibility,* for the nonseparable case. It is shown that conventional ATLE's cannot cope with nonseparability even according to this very weak criterion.**

***Index Terms*—Adaptive threshold logic, convergence, perceptron, nonseparability, threshold.**

I. INTRODUCTION

The simple intuitively meaningful structure of the adaptive threshold logic elements (ATLE's) developed by Rosenblatt [1], Widrow [2], and many others [3], coupled with the availability of straightforward proofs [4] of their convergence under reasonable conditions, has commended them to many workers in the fields of pattern recognition and machine learning. However, Minsky and Papert in their analysis of the computational power of threshold logic [5] have remarked on the irrelevance of the basic convergence proofs to justifying the incremental "learning" strategy of an ATLE. They note, for example, that the random selection of a new solution (weight vector) when the current one fails also "converges," and give examples where such a "convergence" is actually more rapid than that of a conventional ATLE.

This correspondence provides a result on the *stability* of ATLE convergence under conditions of perturbation by a transient sequence of incorrect reinforcement that does bring out a major advantage of incremental "learning." It also clarifies the role of the *"threshold,"* which plays surprisingly little part in the usual convergence proofs.

The nature of the result obtained suggests a new concept of *"admissibility"* of an ATLE "solution" in the nonseparable case, and the final section of this correspondence is concerned with developing this concept.

Manuscript received July 25, 1975; revised January 21, 1976.
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II. RECONVERGENCE OF A PERTURBED ATLE

We consider the behavior of an ATLE that has already converged to a solution of a particular classification and which then receives a perturbation (due to incorrect reinforcement, say) that causes its weight vector to change. We are interested in the possibility of its reconverging to a solution *without making any errors in classifying the inputs,* and show that this is always possible (Result 1) provided its threshold is large enough compared with the perturbation of the weight vector. It is clearly trivial to show this for the weight vector immediately after the perturbation, but less so to demonstrate it for the entire trajectory of weight vectors during reconvergence.

By appropriate redefinition of the pattern vectors [4], the ATLE problem of separating two finite sets of $(n-1)$-dimensional vectors can be stated in terms of a single set, X, to be find an n-dimensional "weight" vector whose scalar product with any number of X is positive, i.e.,

$$x \in X, \qquad W \cdot x > 0. \tag{1}$$

The standard ATLE weight adjustment procedure of changing the weight vector by adding to it the current pattern vector if and only if their scalar product is less than a positive threshold, $\theta > 0$, is also simply represented. Let $x(i)$ and $w(i)$ be the ith element of a training sequence drawn from X and the ith value of the weight vector, respectively, for which a weight adjustment occurs. Then, we have

$$w(i+1) = w(i) + x(i) \tag{2}$$

and

$$w(i) \cdot x(i) < \theta. \tag{3}$$

Note the distinction between the 0 in inequality (1) and the θ in (3) —the weight vector is a solution giving the correct classification if it satisfies (1), but it is still changed if it satisfies (3). This distinction between the criterion of performance and the criterion for adaption is not important in normal convergence proofs where only adaption is considered. We are concerned to show that the performance can be maintained while adaption occurs, and the distinction is of prime importance. The threshold, θ, is a form of "noise margin," and the technique of adapting according to one criterion while basing performance on another is common to many "tracking systems," e.g., in adaptive line equalization [6].

Standard Convergence Proof

We base our later discussion on the standard ATLE convergence proof [4]. Assume there is a W satisfying (1), i.e., the original classes are linearly separable, and that convergence can be proved if the sequence $x(i)$, can be shown to be finite, i.e., all members of X occur frequently in the training sequence from which $x(i)$ is derived. Let us first define two basic parameters of the set X: P is the minimum scalar product of any $x \in X$ with W, and M is the maximum length of any $x \in X$ (both P and M can be normalized to unity without loss of generality). We have

$$W \cdot x(i) \geq P \tag{4}$$

and

$$x(i) \cdot x(i) \leq M^2. \tag{5}$$

Hence, squaring each side of (2) and using (3) and (5):

$$w(i+1) \cdot w(i+1) < w(i) \cdot w(i) + 2\theta + M^2. \tag{6}$$

Summing (5) over $0 \leq i < N$:

$$w(N) \cdot w(N) < w(0) \cdot w(0) + N(2\theta + M^2). \tag{7}$$

This is one basic inequality for the length of $w(N)$.

Now, taking scalar products on both sides of (2) with W and using (4):

$$w(i+1) \cdot W \geq w(i) \cdot W + P. \tag{8}$$

Summing (8) over $0 \leq i < N$:

$$w(N) \cdot W \geq w(0) \cdot W + NP. \tag{9}$$

Hence, using the Schwartz inequality:

$$w(N) \cdot w(N) \geq (w(0) \cdot W + NP)^2 / W \cdot W. \tag{10}$$

This is the second basic inequality for $w(n)$, and clearly must eventually become incompatible with that of (7) as N increases. Hence, the sequence $w(i)$ is finite with the maximum value of i being I, say. Combining (7) and (10), we obtain an inequality for I:

$$(PI + w(0) \cdot W)^2 < W \cdot W(w(0) \cdot w(0) + I(2\theta + M^2)). \tag{11}$$

Adding $(W \cdot W)(2\theta + M^2)((W \cdot W)(2\theta + M^2)/4P^2 - (W(0) \cdot W)/P - I)$ to both sides, multiplying by 4, and factoring, we obtain

$$(2PI + 2w(0) \cdot W - W \cdot W(2\theta + M^2)/P)^2 < W \cdot W|2w(0) - W(2\theta + M^2)/P|^2. \tag{12}$$

Taking the positive square root of both sides we obtain

$$2PI - 2|w(0)||W| - W \cdot W(2\theta + M^2)/P < |W|(2|w(0)| + |W|(2\theta + M^2)/P). \tag{13}$$

So that, finally

$$I < 2|w(0)||W|/P + W \cdot W(2\theta + M^2)/P^2. \tag{14}$$

If we consider the normalized case with $M = P = 1$, commencing from a zero weight vector, $w(0) = 0$, (14) reduces to

$$\text{number of corrections to convergence} \qquad I < W \cdot W(2\theta + 1) \tag{15}$$

where $W \cdot W$, given that W has to satisfy (4), may be seen as a measure of the "difficulty" of a particular cone of vectors X.

Behavior During Reconvergence

Now consider the ATLE converged to some weight vector, V say, so that we have

$$x \in X, \qquad V \cdot x \geq \theta. \tag{16}$$

Suppose V is now perturbed by the addition of some bounded vector U such that

$$U \cdot U \leq B^2 \tag{17}$$

where U represents a disturbance to the weight vector caused, for example, by a sequence of incorrect reinforcements to the ATLE. Intuitively, if V is large and U is small, it may be possible for the ATLE to reconverge without the resulting changes to V being sufficient to lead to errors. We can clearly make V large by increasing the threshold θ, but, from (14) this would also seem to increase the number of corrections to reconvergence. However, the following argument shows that this is not so, and the number of corrections is a function only of B, the bound on the magnitude of U.

We need to make use of the fact that V is already a solution. Note first that (7) and (14) together imply that, if θ is increased, then $|W(N)|$ (and hence $|V|$) grows at most of order θ. Now if we use the equations of the previous section to describe the reconvergence of the ATLE and define W in terms of V as

$$W = VP/\theta, \tag{18}$$

then W is a solution satisfying (4) and $|W|$ is nonincreasing with θ. Now we take $w(0)$ to be the perturbed weight vector

$$w(0) = V + U \tag{19}$$

and substitute this in (12), eliminating V through (18) to obtain an estimate for I, the number of corrections to reconvergence

$$(2PI + 2W \cdot U - W \cdot WM^2/P)^2 < W \cdot W|2U - WM^2/P|^2. \tag{20}$$

Taking positive square roots as before:

$$2PI - 2|W||U| - W \cdot WM^2/P < 2|W||U| + W \cdot WM^2/P. \tag{21}$$

Grouping terms and using (17):

$$I < 2|W|B/P + W \cdot WM^2/P^2. \tag{22}$$

Hence, again for the normalized case $M = P = 1$, we have

$$\text{number of corrections to reconvergence}$$
$$I < 2|W|B + W \cdot W. \tag{23}$$

Note that (22) and (23) are independent of θ.

Now consider any $x \in X$ and take the scalar product of both sides of (2) with x:

$$w(i+1) \cdot x = w(i) \cdot x + x(i) \cdot x. \tag{24}$$

Hence, from (5):

$$w(i+1) \cdot x > w(i) \cdot x - M^2. \tag{25}$$

Summing (25) for $0 \leq i < N$:

$$w(N) \cdot x > w(0) \cdot x - NM^2. \tag{26}$$

Substituting from (19), (16), and (17):

$$w(N) \cdot x > \theta - BM - NM^2. \tag{27}$$

Hence, there will be no errors during reconvergence if

$$\theta > BM + NM^2, \tag{28}$$

but the maximum value of N is I of (22), so that (28) implies

$$\theta > BM + 2|W|BM^2/P + W \cdot WM^4/P^2 \tag{29}$$

which, for the normalized case $M = P = 1$, reduces to

$$\theta > B + 2|W|B + W \cdot W. \tag{30}$$

Inequalities (29) and (30) show clearly that a sufficiently large value of θ can compensate for any size of perturbation to the solution weight vector, and ensure that reconvergence takes place without inducing any errors in misclassification. If the perturbation is due to transient incorrect performance feedback then B will be at most M times the number of false corrections, and hence any period of false feedback whose maximum duration is known in advance can be allowed for by a sufficiently large threshold θ. This may be summarized as follows.

Result 1: An ATLE that has converged to a solution of a classification and is then subject to a perturbation of its weight vector will reconverge without errors of misclassification provided its threshold is sufficiently large compared with the magnitude of the perturbation.

III. Admissibility as a Criterion in the Nonseparable Case

The result of Section II, that a sufficiently high threshold allows an ATLE to reconverge after a perturbation with no errors of misclassification (at the expense, only, of an increased training period), in itself provides a counterexample to the argument in [5] that a random selection of the weight vector when an error occurs is as good as the ATLE's incremental learning strategy. Clearly random selection will lead to reconvergence being exactly the same as convergence and subject to misclassification errors. It is the basic *inertia of incremental learning* that leads to the result established in the previous section.

The concept introduced in Section II, of the *stability* of ATLE performance under perturbation, seems of interest in its own right. In more general terms, the capability of a learning system to modify its behavior without making errors by applying a more stringent criterion to its own behavior than is necessary for the required performance is clearly an interesting phenomenon. As already noted, it is the basis of some adaptive line equalization systems [6], but it also has some obvious cognates in human and animal behavior. We decided to investigate whether it was possible to show that an ATLE could maintain its performance even under nontransient disturbances, e.g., in readapting to a new set of pattern vectors. Defining this concept rigorously is not trivial because in general the union of old and new sets of pattern vectors will be nonseparable and "error-free" performance is not definable. However, there is a distinction between reasonable errors and unnecessary errors, that is some weight vectors will be better than others because they give incorrect results with a smaller subset of pattern vectors.

Our attempts to define this situation have led to the following new treatment of the nonseparable case. This is important in more general terms because one is concerned that, when a learning system cannot converge to error-free performance, at least it behaves reasonably. Although there have been previous studies of the nonseparable case [7], [8], they have not introduced a general definition of convergence in this situation. This we now do in terms of the concept of admissible solutions, borrowed from statistics [9] and used in control theory [10].

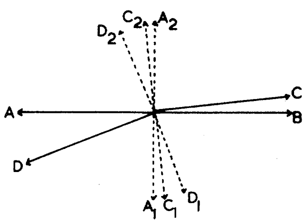

Fig. 1. Counterexample to stable convergence.

Admissible Weight Vectors

We first define a natural preorder relation on all possible weight vectors, in fact one of subset inclusion determined by the subset of X with which a weight vector has positive scalar products.

Definition 1: If X is a set of pattern vectors (normalized in the usual way by sign changing to form a single class rather than two [3]) and W and W' are two weight vectors, then we define a preorder relation on the weight vectors, $\geq$, by

$$W \geq W' \Leftrightarrow x \in X, \quad W' \cdot x > 0 \Rightarrow W \cdot x > 0, \tag{31}$$

i.e., any pattern correctly classified by W' is also correctly classified by W.

We are now in a position to define admissibility.

Definition 2: A weight vector is *admissible* if it is maximal in the preorder relation defined by (31).

Hence, an admissible weight vector has no others properly greater than it in the preorder relation, i.e., such that they correctly classify all those correctly classified by it *plus* some others.

Thus, admissibility provides the weaker criterion necessary in the nonseparable case. We can no longer say that a weight vector is *optimal* in providing a solution, but we can at least say it is *admissible* if there are no better attempted solutions. The separable case now appears as a special condition upon the preorder relation.

Result 2: The set of pattern vectors, X, is linearly separable if and only if all admissible solutions are equivalent in the preorder relation of (31).

Proof: If the set is linearly separable then there is at least one W satisfying, $x \in X, W \cdot x > 0$. From (31), for any other W', we have $W \geq W'$, so that the only admissible solutions must be equivalent to W in the preorder relation. Conversely, if all solutions are equivalent then there is no $x \in X$ which is misclassified by one weight vector and not another. However, x itself is trivially a weight vector correctly classifying x. Hence, there are no x which are misclassified by any admissible solutions. Hence, the set is linearly separable.

In the nonseparable case the ATLE weight vector will not in general cease changing after a finite number of corrections during a training period, but will continue to change *ad infinitum.* However, we can define an appropriate form of "convergence" by requiring its weight vector to become, and remain, admissible. Because of this requirement for continuing admissibility we have called this *stable* convergence.

Definition 3: An ATLE has *stably converged* for a set of pattern vectors X, if and only if, its weight vector is admissible and, given any training sequence drawn from X, its weight vector changes in such a way as to remain admissible.

This concept of *stable convergence* is shown by Result 2 to be a generalization of normal ATLE convergence and equivalent to it in the separable case. It is an important practical extension because separability often cannot be guaranteed and stable convergence would at least mean that no better solutions were available. It also provides a sensible criterion for behavior during readaption from a set X to X' in that the weight vector should be admissible for $X, X \cup X'$, or X' at appropriate times.

Definition 4: An ATLE which has stably converged for a set X, in reconverging with a training sequence drawn from a set X', shows *stable reconvergence* if its weight vector is all the time admissible for at least one of $X, X \cup X'$, or X', finally becoming stably converged for X'.

The Existence of ATLE's Showing Stable Convergence and Reconvergence

It seems reasonable to argue that admissibility and stable convergence provide the very weakest criteria by which one may judge ATLE performance in the nonseparable case, and that stable reconvergence is a sensible design objective for an ATLE required to operate in a changing environment. Unfortunately, simple counterexamples show that the conventional ATLE cannot satisfy these criteria in general.

To provide a general counterexample while avoiding the possibility of the ATLE "remembering" all past patterns input, it is necessary to place some weak constraints upon it which still, however, cover all published algorithms.

Result 3: Stable convergence and reconvergence are impossible in general for an ATLE whose next weight vector is a convex function of its previous weight vector and pattern vector, and which converges in the linearly separable case.

Proof: Consider the planar nonseparable set of vectors, $\{A,B,C,D\}$, shown in Fig. 1. The subscripted weight vectors A_1, A_2, etc., are at right angles to the relevant nonsubscripted vector, and delimit the cones of admissible weight vectors: A_2C_2 for $\{A,C\}$; C_1D_1 for $\{B,C,D\}$; and A_1D_2 for $\{A,D\}$—where, for example, A_2C_2 indicates the convex cone of vectors of the form $\alpha A_2 + \beta C_2$, $\alpha > 0$ and $\beta > 0$.

Now consider the training sequence in which at some arbitrary time a sequence consisting of A's and C's only is presented. Since the ATLE converges in the separable case the weight vector must eventually lie in A_2C_2. At this time the training sequence switches to a sequence of B's only. Again the ATLE will readapt but there is *no* convex combination of B and a vector from A_2C_2 that is both admissible and has a positive projection on B.

This shows that the convergence to the set $\{A,B,C,D\}$ cannot be *stable* in terms of Definition 4 given what is a legitimate training sequence from that set. The same example also shows lack of stable reconvergence from the set $\{A,C\}$ to the set $\{B,D\}$.

IV. Conclusions

It has been shown that an ATLE whose weight vector is perturbed reconverges without error provided its *threshold* is large compared with the perturbation. This result has been generalized to a criterion of convergence in the nonseparable case in which a weight vector is said to be *admissible* if the set of pattern vectors on which it has a positive projection is maximal in that it is not a proper subset of that for some other weight vector.

Stable convergence to a nonseparable set of pattern vectors is then defined as continuing admissibility given any arbitrary training sequences from the set, and *stable reconvergence* from one pattern set to another has been defined on a similar basis. It is shown that neither stable convergence nor stable reconvergence are possible in general for any of the normal range of ATLE's.

These results clarify the role of the threshold as a "noise-margin" in the ATLE, and also the fundamental weakness of the ATLE in the nonseparable case.

References

[1] F. Rosenblatt, *Principles of Neurodynamics.* Washington: Spartan, 1962.

[2] B. Widrow and F. W. Smith, "Pattern-recognizing control systems," in *Computer and Information Sciences,* J. T. Tou and R. H. Wilcox, Eds. Washington: Spartan Books, 1964, pp. 288–317.

[3] A. Novikoff, "On convergence theorems for perceptrons," in *1962 Proc. Symp. on Mathematical Theory of Automata.* (Brooklyn, NY: Polytechnic Press, 1962), pp. 615–622.

[4] N. Nilsson, *Learning Machines.* New York: McGraw-Hill 1965.

[5] M. Minsky and S. Papert, *Perceptrons.* Cambridge, MA: MIT Press, 1969.

[6] R. W. Lucky, "Techniques for the adaptive equalization of digital communication systems," *Bell Syst. Tech. J.,* vol. 45, pp. 255–286, 1966.

[7] B. Efron, "The perceptron correction procedure in non-separable situations," Tech. Doc. Rep. RADC-TDR-63-533, 1964.

[8] A. R. Butz, "Perceptron type learning algorithms in non-separable situations," *J. Math. Anal. and Appl.,* vol. 17, pp. 560–576, 1967.

[9] L. Weiss, *Statistical Decision Theory.* New York: McGraw-Hill, 1961.

[10] H. Kwakernaak, "Admissible adaptive control," in *1965 Proc. IFAC Symp. Theory of Self-Adaptive Control Systems.* (London: Society of Instrument Technology, 1965).

Realization of Fault-Tolerant and Fail-Safe Sequential Machines

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***Abstract*—Given a synchronous sequential machine M, this correspondence deals with the fault-tolerant realization $\overline{M}$ of M and also its fail-safe realization $\hat{M}$ on the assumption that the faults that can occur to the circuitry of $\overline{M}$ or $\hat{M}$ are of permanent stuck-at type and the total number of faults can be at most some preset positive integer r. First, the realization of $\overline{M}$ or $\hat{M}$ is derived for $r = 1$ and then the same idea is extended to have the realization for any arbitrary r. It has been shown that the realization of $\overline{M}$ is such that it is also able to tolerate faults of nonpermanent nature.**

***Index Terms*—Error correcting codes, fail-safe realization, fault masking, replication redundancy, single and multiple fault-tolerance.**

I. Introduction

An r-fault-tolerant realization $\overline{M}$ of a synchronous machine M is that realization of M which produces normal input-output behavior of M even when a number of faults occur to the circuitry of $\overline{M}$, the total number of faults being less than or equal to r, whereas an r-fail-safe realization $\hat{M}$ of M is that realization of M which either continues to produce normal input-output behavior of M or produces a safeside output for all applied inputs whenever a number of faults occur to the circuitry of $\hat{M}$; the total number of faults being less than or equal to r and each fault being of permanent stuck-at type. Evidently, therefore, for reliable operation of a synchronous sequential machine it should be designed incorporating fault-tolerant capabilities. A fail-safe realization $\hat{M}$, on the other hand, needs a check up of its circuitry whenever the safeside output is produced, indicating faults may have occurred to it. Several authors have dealt with the problem of fault-tolerant realization $\overline{M}$ [1]–[5] as well as the fail-safe realization $\hat{M}$ [6]–[7] of a machine M. The r-fault-tolerant realization is derived in these papers using the principle of assigning minimum Hamming distance $2r + 1$ codes to the states of the machine (the principle commonly used in coding theory to design error correcting codes capable of correcting at most r errors [8]) which are obtained either from linear block code or from past states of the machine, etc. On the other hand, the fail-safe realization $\hat{M}$ is derived by designing the machine in such a way that the machine ultimately reaches a terminal state whenever a fault occurs in the circuit of $\hat{M}$ and thus, goes on producing the safeside output. Apparently, therefore, the designs of $\overline{M}$ and $\hat{M}$ are not related to each other at all. In this paper, the fault-tolerant and the fail-safe realizations are derived from the same viewpoint. Depending on the value of r, a number of copies of the machine M is taken and it is then detected whether any of them is faulty or not. If any of them is found to be faulty, then in case of realization of $\overline{M}$, the correct state of M is derived from the states of the copies of M, and in case of realization of $\hat{M}$, the output of all the copies are suppressed by the safeside output. It has been shown that in $\overline{M}$ or $\hat{M}$ even if the machine/machines which detect any fault of the copies of M become faulty, the entire arrangement remains to be respectively, fault-tolerant or fail-safe. This process of replication of M to realize $\overline{M}$ or $\hat{M}$ causes the circuit realization of $\overline{M}$ or $\hat{M}$ to be greatly simplified, although the memory element requirements may be higher than that in case of realization with r error correcting codes.

Throughout this correspondence, it will be assumed that a machine M with n number of memory elements is represented by Fig. 1, where C_i is the excitation circuit of the ith memory element, which is chosen as a unit delay element. The inputs to C_i are the primary input vector and present internal state vector. The block Z is a purely combinational network which generates the output of M from the present state vector and the primary input. As the fault-tolerant and fail-safe realization of combinational networks has been well studied by several authors [9]–[11], this paper will not deal with such realizations of the block Z and will assume such realizations. This is why it is said earlier that to realize $\overline{M}$, it is sufficient to generate the correct state of M from the states of the copies of M. It is shown in this correspondence that $r + 1$ number of copies of M is sufficient to realize $\overline{M}$ to tolerate r number of faults as compared to $2r + 1$ copies as in [12]. In Sections II and III, realizations of $\overline{M}$ and $\hat{M}$, respectively, are discussed for $r = 1$ and extending the same principle, realizations of $\overline{M}$ and $\hat{M}$ for arbitrary r are discussed, respectively, in Sections IV and V.

II. Single Fault-Tolerant Realization of M

From the representation of Fig. 1 of a machine M, it readily follows that if r number of faults occur anywhere in the circuit of the state machine version of M, then at the first time when the state of M deviates from its normal state, the deviation cannot be in more than r bit positions of the corresponding code vector of the state. The deviation even may be in less than r bit positions corresponding to the case when some of the faults occur to the same C_i. In this section, realization of $\overline{M}$ will be dealt with corresponding to $r = 1$. In fact, the realization that will be discussed here applies to the case where due to fault in M, whenever the state of M deviates from its normal state for the first time, the deviation is in only one bit position. Thus, this realization is also able to tolerate some particular type of multiple faults as well, e.g., when only one C_i is faulty, any number of multiple faults in that C_i will be tolerated.

Let F_i denote the logical function generated by C_i and F_i^e denote that when some faults have occurred to C_i. Let F denote the next state function of M. If s_j is the state of M at any instant and it differs from the corresponding normal state of M in p number of bit positions of the corresponding code vector of the state, then it will be denoted as $s_j \sim p$.

Definition 1: An ordered pair of a present state of M and an input symbol is defined as a total state of M.

Each C_i generates a logical function corresponding to all total states of M. Assuming the occurrence of single fault to M, the fault may occur 1) to any of the C_i's, or 2) to the input of any of the delay elements, or 3) to the output of any of the delay elements.

Definition 2: A total state of M will be said to detect a fault of M if for that total state, 1) F_i and F_i^e differ, when the fault occurs to C_i, or 2) F_i and the stuck-at fault of the input of the ith delay element differ when the fault is at the input of the ith delay element, or 3) F_i and the stuck-at fault of the output of the ith delay element differ, when the fault is at the output of the ith delay element. Suppose, at any instant, M is in the state s_j, so that $s_j \sim 0$. Then, for an input symbol I_k, for which the total state (s_j, I_k) does not detect the fault of M, $F(s_j, I_k) \sim 0$. Thus, for any fault of M if at any instant M is at the nondetecting correct total state, M masks the fault itself and the state of M at the next instant is the correct next state of M. Now, if $s_j \sim 0$ and if the input to M is I_k such that (s_j, I_k) detects the fault of M, then on the assumption of single fault, $F(s_j, I_k) \sim 1$, i.e., if at any instant M is at the detecting correct total state, the state of M at the next instant is erroneous and the error is to be detected and correct state at the next instant is to be determined. If the fault of M is such that the initial state $s_j \sim 1$, ($s_j \sim t$, $t > 1$, cannot exist because of single fault occurrence) the correct state at the present instant is to be determined. Thus, to achieve fault-tolerant realization, the correct state of M at the instant is to be determined only if the total state of M at the previous instant detects the fault and this determination is carried on by taking a second copy of M. We shall refer the fault-tolerant realization achieved by this process as the Type I realization.

In the Type II fault-tolerant realization, two copies of M are also taken and the output of one copy is suppressed by that of the other whenever the correct total state of the former detects its fault for the first time (or if the initial state $s_j \sim 1$) and suppression is continued for all future in-

Manuscript received February 13, 1975; revised November 15, 1975.

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